ON THE COMMUTATOR OF UNIT QUATERNIONS AND THE NUMBERS 12 AND 24

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Abstract. The quaternions are non-commutative. The deviation from commutativity is encapsulated in the commutator of unit quaternions. It is known that the k-th power of the commutator is null-homotopic if and only if k is divisible by 12. The main purpose of this paper is to construct a concrete null-homotopy of the 12-th power of the commutator. Subsequently, we construct free $\mathbb{S}^3$-actions on $\mathbb{S}^7 \times \mathbb{S}^3$ whose quotients are exotic 7-spheres and give a geometric explanation for the order 24 of the stable homotopy groups $\pi_{n+3}(\mathbb{S}^n)$. Intermediate results of perhaps independent interest are a construction of the octonions emphasizing the inclusion $\mathrm{SU}(3) \subset \mathrm{G}_2$, a detailed study of Duran's geodesic boundary map construction, and explicit formulas for the characteristic maps of the bundles $\mathrm{G}_2 \to \mathbb{S}^6$ and $\mathrm{Spin}(7) \to \mathbb{S}^7$.

1. Introduction

The quaternions form a non-commutative normed division algebra. Samelson [Sa] and G. W. Whitehead [Wh] proved that the quaternions are not even homotopically commutative. More precisely, the commutator

$$[\,\cdot\,,\,\cdot\,] : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3, \quad (a,b) \mapsto [a,b] := aba^{-1}b^{-1}$$

of unit quaternions generates the homotopy group $\pi_6(\mathbb{S}^3) \approx \mathbb{Z}_{12}$ (see [BS], [Ro], [Ja]). It follows that the k-th power

$$(a,b) \mapsto [a,b]^k = aba^{-1}b^{-1}aba^{-1}b^{-1} \dots aba^{-1}b^{-1}$$

is null-homotopic if and only if k is divisible by 12, and that the map

$$(a,b) \mapsto [a,b^k] = ab^k a^{-1} b^{-k}$$

is null-homotopic if and only if k is divisible by 12 (see [Wh]).

Main Construction. *We construct two concrete homotopies*

$$\mathbb{S}^3 \times \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$$

that deform $[\,\cdot\,,\,\cdot\,]^{12}$ *and* $(a,b) \mapsto [a,b^{12}]$*, respectively, to the constant map to* 1.

The twelve H-space structures on $\mathbb{S}^3$ can be represented by the multiplications $(a,b) \mapsto ab[a,b]^k$ for $k = \{0,\dots,11\}$ (see [AC]). A trivial consequence of our construction is a concrete deformation between $(a,b) \mapsto ab[a,b]^{12}$ and the standard multiplication on $\mathbb{S}^3$. Other, less trivial, consequences are based on the following subsequent construction: Let $\mathrm{Sp}(2)$ denote the group of unitary quaternionic 2×2 matrices and let $p_{1,2} : \mathrm{Sp}(2) \to \mathbb{S}^7$ denote the projection to the first/second column,

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respectively. We fix a suitable identification of $\mathbb{S}^7$ with the unit octonions. Let $\downarrow^j$ denote the selfmap of $\mathbb{S}^7$ that sends each unit octonion to its j-th power.

Construction 1.1. *Using the null-homotopies of the Main Construction we construct maps $\chi_j : \mathbb{S}^7 \to \mathrm{Sp}(2)$ such that $p_1 \circ \chi_j = \downarrow^{12j}$. This yields a concrete identification $\mathbb{Z} \to \pi_7(\mathrm{Sp}(2))$.*

Via the generalized Gromoll-Meyer construction in [DPR] we then obtain concrete exotic free $\mathbb{S}^3$-actions on $\mathbb{S}^7 \times \mathbb{S}^3$. The existence of such actions was established by Hilton and Roitberg in 1968 [HR]. Our construction owes to the reinvestigation of this phenomenon in [BR].

Concretely, let $\chi_{j,2} : \mathbb{S}^7 \to \mathbb{S}^7$ be an abbreviation for $p_2 \circ \chi_j$ and let $\langle\!\langle u, v\rangle\!\rangle = \bar{u}^{\mathrm{t}} v$ denote the standard Hermitian product on the quaternionic vector space $\mathbb{H}^2$. For a unit quaternion $q \in \mathbb{S}^3$, a quaternionic vector $u \in \mathbb{H}^2$, and a unit quaternion $r \in \mathbb{S}^3$ set

$$q \star_j (u, r) = \big(qu\bar{q}, \langle\!\langle \chi_{j,2}(qu\bar{q}), q\chi_{j,2}(u)\rangle\!\rangle r\big).$$

Note that $q \star_0 (u, r) = \big(qu\bar{q}, qr\big)$ since χ_0 is the constant map to the unit matrix in $\mathrm{Sp}(2)$. All other actions $\star_j$ are not isometric with respect to the standard metric on $\mathbb{S}^7 \times \mathbb{S}^3$.

Theorem 1.2. *The action $\star_j$ is a free $\mathbb{S}^3$-action on $\mathbb{S}^7 \times \mathbb{S}^3$ whose quotient space is the homotopy 7-sphere Σ^7_{12j} in the notation of* [DPR]*. All these homotopy 7-spheres form a subgroup isomorphic to $\mathbb{Z}_7$ in the group of orientation preserving diffeomorphism classes of homotopy 7-spheres $\Theta_7 \approx \mathbb{Z}_{28}$.*

Another application concerns the stable homotopy groups $\pi_{n+3}(\mathbb{S}^n)$, $n \geq 5$. It is well-known (see e.g. [Hu]) that these groups are cyclic of order 24 generated by the suspensions $\Sigma^{n-4}h$ of the Hopf map

$$h : \mathbb{S}^7 \to \mathbb{S}^4, \quad \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} |u|^2 - |v|^2 \\ 2\bar{u}v \end{smallmatrix}\right).$$

Here, u, v are quaternions with $|u|^2 + |v|^2 = 1$.

Construction 1.3. *Employing the maps $\chi_j : \mathbb{S}^7 \to \mathrm{Sp}(2)$ we obtain simple explicit homotopies between $\Sigma(h \circ \downarrow^{12j})$ and its natural homotopy inverses $\Sigma^{-1}(h \circ \downarrow^{12j})$ given by reversing the direction of the suspension coordinate.*

In fact, this construction is short enough to perform it right away: The map

$$H_j : \mathbb{S}^7 \times [0, \tfrac{\pi}{2}] \to \mathbb{S}^7, \quad H_j(x, t) = \cos t \cdot \chi_{j,1}(x) + \sin t \cdot \chi_{j,2}(x).$$

yields a well-defined homotopy between $\chi_{j,1} = \downarrow^{12j}$ and $\chi_{j,2}$ since corresponding values of $\chi_{j,1}$ and $\chi_{j,2}$ are always perpendicular in $\mathbb{H}^2$. Now,

$$h\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) = \left(\begin{smallmatrix} |c|^2 - |d|^2 \\ 2c\bar{d} \end{smallmatrix}\right) = \left(\begin{smallmatrix} |b|^2 - |a|^2 \\ -2a\bar{b} \end{smallmatrix}\right) = -h\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \quad \text{if } \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{Sp}(2). \tag{1}$$

Hence, $h \circ H_j$ is a homotopy between $h \circ \downarrow^{12j}$ and $-h \circ \downarrow^{12j}$. This homotopy induces a homotopy $\Sigma(h \circ H_j)$ between the suspensions $\Sigma(h \circ \downarrow^{12j})$ and $\Sigma(-h \circ \downarrow^{12j})$. Now, $\mathrm{id}_{\mathbb{S}^5}$ is homotopic to $-\mathrm{id}_{\mathbb{S}^5}$ by a block matrix that consists of three 2×2 rotation

matrices. The concatenation of the homotopies deforms $\Sigma(h \circ {\downarrow}^{12j}) : \mathbb{S}^8 \to \mathbb{S}^5$ to the map

$$- \operatorname{id} \circ \Sigma(-h \circ {\downarrow}^{12j}) = \Sigma^{-1}(h \circ {\downarrow}^{12j}).$$

The organization of the paper is explained at the end of the next section, which provides an outline of our Main Construction.

2. Outline of the Main Construction

The rough combinatorics of our construction is perhaps not surprising: $12 = 3 \cdot 2 \cdot 2$ where the factor 3 is geometrically related to the order of the homotopy group $\pi_6(\mathrm{G}_2)$, one factor 2 comes from killing the double of the fourth suspension of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$, and the other factor 2 from killing the double of the single suspension of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$. The detailed deformations, however, are far from beeing obvious.

The crucial geometric tool is a particular form of the boundary map in the exact homotopy sequence of fiber bundles. This particular form was first used by Duran [Du]. Given a fiber bundle $F \cdots E \to B$ and a map $\alpha : \mathbb{S}^k \to B$ one uses horizontal lifts to define a concrete map $\partial_{E \to B}(\alpha) : \mathbb{S}^{k-1} \to F$ that induces the boundary map $\pi_k(B) \to \pi_{k-1}(F)$. We investigate this construction in connection with suspensions of maps and powers of spheres and obtain some statements that are specific to this way of lifting.

We now outline our Main Construction. The first step is rather elementary.

Construction 2.1. *We construct a continuous map $\mu : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^6$ and two homotopies $\mathbb{S}^3 \times \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ that deform $[\ \cdot\ ,\ \cdot\ ]^{12}$ and $(a,b) \mapsto [a, b^{12}]$, respectively, to the composition $\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^{12} \circ \mu$.*

Thus it remains to construct a null-homotopy of the 12-th power of the characteristic map $\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})$. Note that a homotopic rational version of this characteristic map was proven to generate $\pi_6(\mathbb{S}^3)$ by Borel and Serre [BS].

The central second step is an application of our results about the horizontal lifting construction in section 4. We apply them to the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r] \arrow[d] & \mathrm{G}_2 \arrow[r] \arrow[d] & V_{7,2} \arrow[d, equal] \\
\mathrm{Sp}(2) \arrow[r] \arrow[d] & \mathrm{Spin}(7) \arrow[r] \arrow[d, "\pi"] & V_{7,2} \\
\mathbb{S}^7 \arrow[r, equal] & \mathbb{S}^7 &
\end{tikzcd}$$

Identify $\mathbb{S}^7$ with the unit sphere in the octonions $\mathbb{O}$ and let $V_{7,2}$ denote the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^7 \approx \operatorname{Im}\mathbb{O}$. Let e_1, e_2 denote a fixed orthonormal 2-frame in $\mathbb{R}^7 \approx \operatorname{Im}\mathbb{O}$. Set

$$\kappa : \mathbb{S}^7 \to V_{7,2}, \quad a \mapsto (\bar{a}e_1a, \bar{a}e_2a).$$

Theorem 2.2. *We have*

$$\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^6 = \partial_{\mathrm{G}_2\to V_{7,2}}(\kappa\circ\downarrow^2),$$

where $\downarrow^2$ *denotes the octonionic squaring map.*

This identity is obtained from the identity $\big(\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id})\big)^6 = \mathbb{1}$, which in turn is intimately related to the geometric presentation of $\pi_6(\mathrm{G}_2)\approx\mathbb{Z}_3$ discovered by Chaves and Rigas [CR1].

The third step is the most technical one. Toda, Saito, and Yokota [TSY] showed that the map κ generates the homotopy group $\pi_7(V_{7,2})\approx\mathbb{Z}_4$ (see also [CR2] for relations between κ and certain Hopf maps). Note that the exact homotopy sequence of the fibration $\mathbb{S}^5\to V_{7,2}\to\mathbb{S}^6$ contains the part

$$\mathbb{Z}_2\approx\pi_7(\mathbb{S}^5)\to\pi_7(V_{7,2})\to\pi_7(\mathbb{S}^6)\approx\mathbb{Z}_2.$$

Thus, the first column of the map $\kappa\circ\downarrow^2$ is null-homotopic.

Construction 2.3. *We construct a concrete homotopy*

$$\kappa\circ\downarrow^2\sim(N,\Sigma^2\tau).$$

Here, N denotes the constant map from $\mathbb{S}^7$ to the northpole e_1 of $\mathbb{S}^6$ and $\Sigma^2\tau$ is the double suspension of a specific map $\tau:\mathbb{S}^5\to\mathbb{S}^3$.

The construction of this homotopy involves two steps. We first recognize that the map κ actually consists of two perpendicular variants of the fibration $\mathbb{S}^7\to\mathbb{CP}^3$ composed with the cut locus collapse $\mathbb{CP}^3\to\mathbb{S}^6$. We deform κ concretely to a map h that consists of two perpendicular variants of the fourth suspension of the Hopf fibration $h_1:\mathbb{S}^3\to\mathbb{S}^2$. In the second step we deform $h\circ\downarrow^2$ to the map $(N,\Sigma^2\tau)$. It is essential here that the second column is the suspension of a map. The precise form of τ is actually not important, yet an explicit formula for this map could be obtained.

Construction 2.4. *Lifting the deformation curves of the homotopy in Construction 2.3 horizontally yields a homotopy*

$$\partial_{\mathrm{G}_2\to V_{7,2}}(\kappa\circ\downarrow^2)\sim A_0\cdot\partial_{\mathrm{G}_2\to V_{7,2}}(N,\Sigma^2\tau)$$

where A_0 is a specific matrix in $\mathrm{SU}(3)\subset\mathrm{G}_2$ that identifies the fiber over the point $(e_1,-e_2)$ with the fiber over the point (e_1,e_2).

For the intuition of the reader we now supply the following commutative diagram

$$\begin{tikzcd}
\mathrm{SU}(2) \arrow[r, equal] \arrow[d] & \mathbb{S}^3 \arrow[d] & \\
\mathrm{SU}(3) \arrow[r] \arrow[d] & \mathrm{G}_2 \arrow[r] \arrow[d, "\pi"] & \mathbb{S}^6 \arrow[d, equal] \\
\mathbb{S}^5 \arrow[r] & V_{7,2} \arrow[r] & \mathbb{S}^6.
\end{tikzcd}$$

Lemma 2.5.

$$\partial_{\mathrm{G}_2\to V_{7,2}}(N,\Sigma^2\tau)=\partial_{\mathrm{SU}(3)\to\mathbb{S}^5}(\Sigma^2\tau).$$

We then apply the Eckmann-Kervaire identity $\partial(\alpha\circ\Sigma\beta)=\partial(\alpha)\circ\beta$, which also holds for the specific way of lifting that we use in this paper (see Lemma 4.5).

Lemma 2.6.

$$\partial_{\mathrm{SU}(3)\to\mathbb{S}^5}(\Sigma^2\tau)=\partial_{\mathrm{SU}(3)\to\mathbb{S}^5}(\mathrm{id}_{\mathbb{S}^5})\circ\Sigma\tau.$$

The map $\partial_{\mathrm{SU}(3)\to\mathbb{S}^5}(\mathrm{id})$ is the suspension of the Hopf fibration $\mathbb{S}^3\to\mathbb{S}^2$. Null-homotopies of twice this map are classical. The simplest null-homotopy $H_{\mathrm{SU}(3)\to\mathbb{S}^5}$ of $\big(A_0\cdot\partial_{\mathrm{SU}(3)\to\mathbb{S}^5}(\mathrm{id})\big)^2$ in precisely this form is given in [PR] (see Theorem 5.3).

Theorem 2.7. *The concatenation of the null-homotopy $H_{\mathrm{SU}(3)\to\mathbb{S}^5}$ with the homotopies of the previous statements provides a null-homotopy*

$$\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^{12}\sim\big(A_0\cdot\partial_{\mathrm{SU}(3)\to\mathbb{S}^5}(\mathrm{id})\big)^2\circ\Sigma\tau\sim 1.$$

The rest of the paper is organized as follows: In section 3 we provide a definition for the octonionic multiplication based on the complex cross product. For our concrete computations in the following sections this definition is much more convenient then the standard definition. In section 4 we study the interplay of suspensions, powers of spheres and Duran's form of the boundary map in the exact homotopy sequence. In section 5 we compute the characteristic maps of bundles that belong to some transitive actions on spheres using horizontal lifts. Moreover, we use the results of section 4 to prove Theorem 2.2. In section 6 we perform Construction 2.1 and in section 7 we perform Construction 2.3 and Construction 2.4. In section 8 we perform the Construction 1.1 and Theorem 1.2.

3. Octonionic multiplication via the complex cross product

We define the *complex cross product* of two vectors $z,w\in\mathbb{C}^3$ by

$$z\times w=\begin{pmatrix}\bar z_2\bar w_3-\bar z_3\bar w_2\\ \bar z_3\bar w_1-\bar z_1\bar w_3\\ \bar z_1\bar w_2-\bar z_2\bar w_1\end{pmatrix}.$$

Let $\langle\!\langle z,w\rangle\!\rangle=\bar z_1w_1+\bar z_2w_2+\bar z_3w_3$ denote the hermitian inner product on $\mathbb{C}^3$. Given two vectors $\begin{pmatrix}z_0\\ z\end{pmatrix}$, $\begin{pmatrix}w_0\\ w\end{pmatrix}\in\mathbb{C}\times\mathbb{C}^3$ we set

$$\begin{pmatrix}z_0\\ z\end{pmatrix}\cdot\begin{pmatrix}w_0\\ w\end{pmatrix}:=\begin{pmatrix}z_0w_0-\langle\!\langle z,w\rangle\!\rangle\\ \bar z_0w+w_0z+z\times w\end{pmatrix}.$$

Theorem 3.1. *With this product, $\mathbb{C}\times\mathbb{C}^3$ is isomorphic to the octonions.*

The proof will be a consequence of the following properties of the complex cross product, all of which can be verified easily.

Lemma 3.2. *For all $y,z,w\in\mathbb{C}^3$ we have*

$$\langle\!\langle z,z\times w\rangle\!\rangle=0\quad\textit{and}\quad y\times(z\times w)=\langle\!\langle y,w\rangle\!\rangle z-\langle\!\langle y,z\rangle\!\rangle w.$$

Lemma 3.3. *If $z,w\in\mathbb{C}^3$ are unit vectors with $\langle\!\langle z,w\rangle\!\rangle=0$ then $z\times w$ is the unique vector in $\mathbb{C}^3$ such that the complex 3×3-matrix $(z,w,z\times w)$ is contained in* $\mathrm{SU}(3)$.

Corollary 3.4. *For all $A \in \mathrm{SU}(3)$ and $z, w \in \mathbb{C}^3$ we have*

$$(A \cdot z) \times (A \cdot w) = A \cdot (z \times w).$$

Lemma 3.5. *For all $z, w \in \mathbb{C}^3$ we have*

$$|z \times w|^2 = |z|^2 |w|^2 - |\langle\!\langle z, w \rangle\!\rangle|^2.$$

Proof of Theorem 3.1. Given an equation of form

$$\left(\begin{smallmatrix} a_0 \\ a \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} w_0 \\ w \end{smallmatrix}\right)$$

we multiply both sides by $\left(\begin{smallmatrix} \bar{a}_0 \\ -a \end{smallmatrix}\right)$ from the left. Using the previous lemmas we obtain

$$(|a_0|^2 + |a|^2) \left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} \bar{a}_0 \\ -a \end{smallmatrix}\right) \left(\begin{smallmatrix} w_0 \\ w \end{smallmatrix}\right).$$

Hence, $\frac{1}{|a_0|^2 + |a|^2} \left(\begin{smallmatrix} \bar{a}_0 \\ -a \end{smallmatrix}\right)$ is the unique left inverse of $\left(\begin{smallmatrix} a_0 \\ a \end{smallmatrix}\right) \neq 0$. Similarly we see that it is also the unique right inverse. Using Lemma 3.5 a straightforward computation shows

$$\left| \left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} w_0 \\ w \end{smallmatrix}\right) \right|^2 = (|z_0|^2 + |z|^2)(|w_0|^2 + |w|^2).$$

All in all, $\mathbb{C} \times \mathbb{C}^3$ is a normed division algebra. □

Remark 3.6. The two maximal subgroups of the automorphism group G_2 of the octonions are $\mathrm{SU}(3)$ and $\mathrm{SO}(4)$. The inclusion $\mathrm{SU}(3) \subset \mathrm{G}_2$ is emphasized by our construction of the octonions via the complex cross product. The inclusion $\mathrm{SO}(4) \subset \mathrm{G}_2$, on the other hand, is emphasized in the standard construction of the octonions using pairs of quaternions with the product

$$(u_1, v_1) \cdot (u_2, v_2) = (u_1 u_2 - \bar{v}_2 v_1, v_2 u_1 + v_1 \bar{u}_2).$$

The inclusion $\mathrm{SO}(4) \subset \mathrm{G}_2$ is then given by the action $(q_1, q_2) \cdot (u, v) = (q_1 u \bar{q}_1, q_2 v \bar{q}_1)$ of two unit quaternions on the octonion (u, v). A concrete isomorphism between the two realizations of the octonions is given by

$$\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_0 + i y_0 \\ x + i y \end{smallmatrix}\right) \mapsto (x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}, y_0 + y_1 \mathbf{i} + y_2 \mathbf{j} + y_3 \mathbf{k}).$$

4. Horizontal lifts, suspensions, and powers of spheres

In this section we study the interplay of suspensions, powers of spheres and the horizontal lifting form of the boundary map in the exact homotopy sequence.

4.1. **Powers of spheres and suspensions.** The *k-th power of the sphere* $\mathbb{S}^n \subset \mathbb{R} \times \mathbb{R}^n$ is in polar coordinates defined by

$$\downarrow^k \left(\begin{smallmatrix} \cos t \\ v \sin t \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos kt \\ v \sin kt \end{smallmatrix}\right).$$

This definition yields real analytic maps $\mathbb{S}^n \to \mathbb{S}^n$ that generalize the algebraically defined k-th powers of the unit spheres in the normed division algebras.

Remark 4.1. The degree of $\downarrow^k$ is k if n is odd. If n is even, the degree of $\downarrow^k$ is 1 if k is odd and 0 if k is even. This is classical. For a generalization of the k-powers to cohomogeneity one manifolds and a unified computation of the degree see [Pü2].

Given a map $\rho : \mathbb{S}^{n-1} \to \mathbb{S}^{m-1}$, view $\mathbb{S}^n \subset \mathbb{R} \times \mathbb{R}^n$ and $\mathbb{S}^m \subset \mathbb{R} \times \mathbb{R}^m$ as suspensions of $\mathbb{S}^{n-1}$ and $\mathbb{S}^{m-1}$ and let

$$\Sigma\rho : \mathbb{S}^n \to \mathbb{S}^m, \quad \left(\begin{smallmatrix} x \\ v \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x \\ |v|\rho(v/|v|) \end{smallmatrix}\right)$$

denote the *suspension* of ρ.

Lemma 4.2. *We have the trivial identity*

$$(\Sigma\rho) \circ \downarrow^k \; = \; \downarrow^k \circ \Sigma\rho.$$

If $m = 2\ell$ and $k = 2j$ are even then the degree of the map $\downarrow^k$ on the right hand side is zero. Hence, $(\Sigma\rho) \circ \downarrow^k$ is null-homotopic. We describe an explicit null-homotopy in the following. Set

$$H_1 : \mathbb{S}^n \times [0,1] \to \mathbb{S}^{2\ell}, \quad \left(\left(\begin{smallmatrix} \cos t \\ v \sin t \end{smallmatrix}\right), s\right) \mapsto \begin{cases} \left(\begin{smallmatrix} \cos 2jt \\ \rho(v) \sin 2jt \end{smallmatrix}\right), & \text{for } t \leq \pi/2, \\ \left(\begin{smallmatrix} \cos 2jt \\ A(s)\cdot\rho(v) \sin 2jt \end{smallmatrix}\right), & \text{for } t \geq \pi/2. \end{cases}$$

Here, $A(s) \in \mathrm{SO}(2\ell)$ is a path from $\mathbb{1}$ to $-\mathbb{1}$, for example given by ℓ copies of a standard 2×2 rotation matrix. The homotopy H_1 deforms $(\Sigma\rho) \circ \downarrow^{2j}$ to the map

$$\left(\begin{smallmatrix} \cos t \\ v \sin t \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \cos 2jt \\ \rho(v)\,|\sin 2jt| \end{smallmatrix}\right).$$

Now set

$$H_2 : \mathbb{S}^n \times [0,1] \to \mathbb{S}^{2\ell}, \quad \left(\left(\begin{smallmatrix} \cos t \\ v \sin t \end{smallmatrix}\right), s\right) \mapsto \left(\begin{smallmatrix} x(s,t) \\ \rho(v)\sqrt{1 - x(s,t)^2} \end{smallmatrix}\right)$$

where $x(s,t) = s + (1-s)\cos 2jt$. The following is now immediate.

Lemma 4.3. *The concatenation of the homotopies H_1 and H_2 deforms $(\Sigma\rho) \circ \downarrow^{2j}$ to the constant map to the north pole of $\mathbb{S}^{2\ell}$.*

Similarly, for even $m = 2l$ and odd $k = 2j+1$ one can explicitly deform $(\Sigma\rho) \circ \downarrow^{2j+1}$ to $\Sigma\rho$.

4.2. **Horizontal lifts.** We now explore Duran's specific form of the boundary map in the exact homotopy sequence of a smooth fiber bundle $F \cdots E \xrightarrow{\pi} B$. Assume that E is equipped with an Ehresmann connection[1]. Suppose we are given a map $\alpha : \mathbb{S}^k \to B$. Let $N = (1, 0, \ldots, 0)$ denote the north pole of $\mathbb{S}^k \subset \mathbb{R}^{k+1}$, let $\mathbb{S}^{k-1}$ be the set of all vectors $v \in T_N\mathbb{S}^k$ with $|v| = 1$, and let γ_v denote the geodesic of $\mathbb{S}^k$ with $\gamma_v(0) = N$ and $\dot\gamma_v(0) = v$. Fix a point $p \in E$. Lift the curve $\alpha \circ \gamma_v$ in B horizontally to a curve $\widetilde{\alpha \circ \gamma_v}$ in E with $\widetilde{\alpha \circ \gamma_v}(0) = p$. Since $\gamma_v(\pi)$ is the south pole S of $\mathbb{S}^k$ the end point $\widetilde{\alpha \circ \gamma_v}(\pi)$ of the lifted curve is contained in the fiber $F_{|\alpha(S)} = \pi^{-1}(\alpha(S))$ for any unit tangent vector v. Set

$$\partial(\alpha) : \mathbb{S}^{k-1} \to F_{|\alpha(S)} \subset E, \qquad v \mapsto \widetilde{\alpha \circ \gamma_v}(\pi).$$

[1]An Ehresmann connection is a complete horizontal distribution. In all our examples we fix a Riemannian metric on the compact manifold E such that $E \to B$ is a Riemannian submersion for some Riemannian metric on B. The horizontal space at a point $x \in E$ is then given by the orthogonal complement of the tangent space to the fiber F_x. The Riemannian metric on B is irrelevant in our following constructions.

Lemma 4.4 (see [Du])**.** *The assignment $\alpha \mapsto \partial(\alpha)$ induces the boundary map*

$$\pi_k(B) \to \pi_{k-1}(F)$$

in the exact homotopy sequence of the fibration $F \cdots E \to B$.

Proof. This is one of the many equivalent topological constructions of the boundary map, see e. g. [Bd], page 452 for an appropriate reference. The specific point here is that the lifting is performed horizontally. $\square$

The Eckmann-Kervaire identity (see [Ke1]) holds also for our specific way of lifting:

Lemma 4.5. *We have*

$$\partial(\alpha \circ \Sigma\beta) = \partial(\alpha) \circ \beta.$$

Proof. It is straightforward to verify that $\alpha \circ \Sigma\beta \circ \gamma_v(t) = \alpha \circ \gamma_{\beta(v)}$. $\square$

Now consider the case where a connected compact Lie group $G \subset \mathrm{SO}(m)$ acts transitively on a manifold B. Suppose that $\mathrm{SO}(m)$ is equipped with a biinvariant Riemannian metric. This induces a biinvariant metric on the subgroup G. Let $\mathbb{1}$ denote the unit element in G, set $p = \pi(\mathbb{1})$, and let $H = G_p$ denote the isotropy group at p. Then $H \cdots G \to B$ is a principal fiber bundle and we can apply the construction above. Assume moreover that a subgroup $K \subset G$ also acts transitively on B. The isotropy group K_p is equal to $K \cap H$. We have a commutative diagram

$$\begin{tikzcd}
K \cap H \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & H/(K \cap H) \arrow[d, equal] \\
K \arrow[r] \arrow[d, "\pi"] & G \arrow[r] \arrow[d, "\pi"] & G/K \\
B \arrow[r, equal] & B &
\end{tikzcd}$$

where the equality $H/(K \cap H) = G/K$ holds since $H/(K \cap H)$ is included in G/K and both spaces have the same dimension. Note that the base manifold B usually inherits different left invariant Riemannian metrics from K and G by Riemannian submersion but this is irrelevant for the lifting construction.

Lemma 4.6. *If $\gamma : [0, T] \to B$ is a curve with $\gamma(0) = \pi(\mathbb{1})$ and $\tilde\gamma^K$, $\tilde\gamma^G$ are the unique horizontal lifts of γ with $\tilde\gamma^K(0) = \tilde\gamma^G(0) = \mathbb{1}$ then*

$$\tilde\delta(t) = \tilde\gamma^G(t)^{-1} \cdot \tilde\gamma^K(t)$$

defines a curve in the fiber $\pi^{-1}(\pi(\mathbb{1})) = H$ and this curve is horizontal with respect to the Riemannian submersion $H \to H/(K \cap H)$.

Proof. Since $\tilde\gamma^K(t)$ and $\tilde\gamma^G(t)$ are lifts of γ, we have $\gamma(t) = \tilde\gamma^K(t) \cdot p = \tilde\gamma^G(t) \cdot p$. Hence $\delta(t) \in H$. Moreover,

$$\dot{\tilde\delta}(t) = -\tilde\gamma^G(t)^{-1} \cdot \dot{\tilde\gamma}^G(t) \cdot \tilde\gamma^G(t)^{-1} \cdot \tilde\gamma^K(t) + \tilde\gamma^G(t)^{-1} \cdot \dot{\tilde\gamma}^K(t)$$

and

$$\tilde{\delta}(t)^{-1} \cdot \dot{\tilde{\delta}}(t) = -\tilde{\delta}(t)^{-1} \cdot \tilde{\gamma}^G(t)^{-1} \cdot \dot{\tilde{\gamma}}^G(t) \cdot \tilde{\delta}(t) + \tilde{\gamma}^K(t)^{-1} \cdot \dot{\tilde{\gamma}}^K(t).$$

Since $\tilde{\gamma}^K$ is horizontal, the second summand is perpendicular to the Lie algebra $\mathfrak{k}\cap\mathfrak{h}$ of $K \cap H$. Since $\tilde{\delta}(t) \in H$ and $\tilde{\gamma}^G$ is horizontal, the first summand is perpendicular to the Lie algebra $\mathfrak{h}$ of H. Hence, $\tilde{\delta}(t)^{-1} \cdot \dot{\tilde{\delta}}(t)$ is perpendicular to $\mathfrak{h} \cap \mathfrak{k}$ and $\tilde{\delta}$ is horizontal. $\square$

Corollary 4.7. *For any map $\alpha : \mathbb{S}^k \to B$ the two maps $\partial_{G\to B}(\alpha)$ and $\partial_{K\to B}(\alpha)$ are homotopic within $\pi^{-1}(\alpha(S)) \approx H$ by the homotopy*

$$(v, s) \mapsto \widetilde{\alpha \circ \gamma_v}^G(\pi) \cdot \widetilde{\alpha \circ \gamma_v}^G(s)^{-1} \cdot \widetilde{\alpha \circ \gamma_v}^K(s).$$

Here, $S = (-1, 0, \dots, 0)$ denotes the south pole of $\mathbb{S}^k$.

Assume now more specifically that the compact Lie group $G \subset \mathrm{SO}(n+1)$ acts transitively on $\mathbb{S}^n$. The isotropy group G_N of the north pole N is denoted by H. As above let γ_v denote the geodesic of $\mathbb{S}^n$ with $\gamma_v(0) = N$ and $\dot{\gamma}_v(0) = v$, $|v| = 1$. Clearly $\gamma_v(t+\pi) = -\gamma_v(t)$ and $\dot{\gamma}_v(k\pi) = (-1)^k \cdot v$. Note that the biinvariant metric on G does not necessarily induce a constant curvature metric on $\mathbb{S}^n$ via Riemannian submersion $H \cdots G \to \mathbb{S}^n$. Hence, γ_v should actually just be called a curve. Let $\tilde{\gamma}_v$ be the unique horizontal lift of γ_v with $\tilde{\gamma}_v(0) = \mathbb{1}$. Then $\dot{\tilde{\gamma}}_v(0) = (v, *)$ is a matrix in the Lie algebra $\mathfrak{g}$ of G whose first column is v.

Lemma 4.8. *We have $\tilde{\gamma}_v(t + \pi) = \tilde{\gamma}_v(t) \cdot \tilde{\gamma}_v(\pi)$, in particular, $\tilde{\gamma}_v(k\pi) = \tilde{\gamma}_v(\pi)^k$.*

Proof. Since $\tilde{\gamma}_v$ is horizontal, $\tilde{\gamma}_v(t)^{-1} \cdot \dot{\tilde{\gamma}}_v(t)$ is perpendicular to the Lie algebra $\mathfrak{h}$ of H. Now consider the curve

$$\sigma(t) = \tilde{\gamma}_v(t) \cdot \tilde{\gamma}_v(\pi).$$

Since

$$\sigma(t) = (\gamma_v(t), *) \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & * \end{smallmatrix}\right) = (-\gamma_v(t), *) = (\gamma_v(t + \pi), *),$$

the curve $\sigma(t)$ is a lift of $\gamma_v(t + \pi)$ with $\sigma(0) = \gamma_v(\pi)$. Moreover,

$$\sigma(t)^{-1} \cdot \dot{\sigma}(t) = \tilde{\gamma}_v(\pi)^{-1} \cdot \big(\tilde{\gamma}_v(t)^{-1} \cdot \dot{\tilde{\gamma}}_v(t)\big) \cdot \tilde{\gamma}_v(\pi).$$

Since $\gamma_v(\pi) = \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$ and the isotropy group at $\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$ is also H, the lift $\tilde{\gamma}_v(\pi)$ normalizes H. Hence, σ is horizontal. By uniqueness, $\sigma(t) = \tilde{\gamma}_v(t + \pi)$. $\square$

Corollary 4.9. *We have*

$$\partial(\downarrow^k) = \partial(\mathrm{id}_{\mathbb{S}^n})^k.$$

where $\downarrow^k : \mathbb{S}^n \to \mathbb{S}^n$ denotes the k-th power of $\mathbb{S}^n$.

Lemma 4.10. *The characteristic map $\partial(\mathrm{id}_{\mathbb{S}^n}) : \mathbb{S}^{n-1} \to G_S = H$ is equivariant with respect to the H action on $\mathbb{S}^{n-1}$ and the adjoint action on H.*

Proof. The curves $\tilde{\gamma}_{hv}(t)$ and $h\tilde{\gamma}_v(t)h^{-1}$ are both horizontal, the first one by definition, the second one because the isotropy group G_{hp} is equal to hG_ph^{-1}. Both curves start at the unit matrix $\mathbb{1}$ and have initial velocity hv. Hence they are identical. Evaluation at time π yields the statement. $\square$

5. Identification of some characteristic maps

In this section we derive explicit formulas for the characteristic maps of the principal bundles

$$\mathrm{SU}(n)\cdots\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1},$$
$$\mathrm{Sp}(n)\cdots\mathrm{Sp}(n+1)\to\mathbb{S}^{4n+3},$$
$$\mathrm{SU}(3)\cdots\mathrm{G}_2\to\mathbb{S}^{6},\quad\text{and}\quad\mathrm{G}_2\cdots\mathrm{Spin}(7)\to\mathbb{S}^{7}$$

using the horizontal lifting construction of section 4. We identify our maps with maps previously given in the literature in other contexts. Finally, we prove Theorem 2.2. Note that the characteristic maps of the principal bundles

$$\mathrm{SO}(n+1)\to\mathbb{S}^{n},\quad \mathrm{U}(n+1)\to\mathbb{S}^{2n+1},\quad \mathrm{Sp}(n+1)\to\mathbb{S}^{4n+3}$$

were already computed before 1950 by a different method, namely, by analyzing how the bundle is glued from two local trivializations over the equatorial sphere (see [St]). Both methods produce homotopic maps. Using horizontal lifts, however, these maps come in a form that is much more suitable for our purposes. In particular, they obey Corollary 4.9. Moreover, the special unitary case was not considered in [St] and in this case we obtain the null-homotopy $H_{\mathrm{SU}(3)\to\mathbb{S}^5}$ that is essential in our main construction.

5.1. **The characteristic map of the principal bundle** $\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1}$**.** The characteristic map $\partial_{\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1}}(\mathrm{id})$ generates the homotopy group

$$\pi_{2n}(\mathrm{SU}(n))\approx\mathbb{Z}_{n!}.$$

(see [Bt] for the computation of this homotopy group). This follows easily by filling in only stable homotopy groups into the exact homotopy sequence of the bundle. In this subsection we identify $\partial_{\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1}}(\mathrm{id})$ with a map ϕ given in [PR] and obtain the homotopy $H_{\mathrm{SU}(3)\to\mathbb{S}^5}$ of Theorem 2.7.

Let $\mathrm{SU}(n+1)$ denote the group of all complex $(n+1)\times(n+1)$-matrices A with $\bar{A}^{\mathrm{t}}A=\mathbb{1}$. The group $\mathrm{SU}(n+1)$ acts transitively on the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$. We endow $\mathrm{SU}(n+1)$ with the up to scaling unique biinvariant Riemannian metric. We consider the curve

$$\gamma(t)=\cos t\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+\sin t\left(\begin{smallmatrix}iy\\z\end{smallmatrix}\right)$$

in $\mathbb{S}^{2n+1}$ with $y\in\mathbb{R}$, $z\in\mathbb{C}^n$, and $|y|^2+|z|^2=1$. For $z\neq0$ we set

$$\tilde\gamma(t)=\left(\begin{pmatrix}0&0\\0&\mathbb{1}-\frac{z}{|z|}\frac{\bar z^{\mathrm{t}}}{|z|}\end{pmatrix}+\cos t\begin{pmatrix}1&0\\0&\frac{z}{|z|}e^{iyt}\frac{\bar z^{\mathrm{t}}}{|z|}\end{pmatrix}\right.$$
$$\left.+\sin t\begin{pmatrix}iy&-e^{iyt}\bar z^{\mathrm{t}}\\z&-\frac{z}{|z|}iye^{iyt}\frac{\bar z^{\mathrm{t}}}{|z|}\end{pmatrix}\right)\cdot\begin{pmatrix}1&0\\0&e^{-iyt/n}\,\mathbb{1}\end{pmatrix}.$$

Lemma 5.1. *The curve $\tilde\gamma$ is the unique horizontal lift of γ with respect to the projection $\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1}$ such that $\tilde\gamma(0)=\mathbb{1}$. In particular, the above formula for $\tilde\gamma$ extends analytically to the case where $z=0$.*

Proof. It is evident that $\tilde{\gamma}(t)\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)=\gamma(t)$. Hence, $\tilde{\gamma}$ is a lift of γ. A direct computation shows that

$$\tilde{\gamma}(t)^{-1}\cdot\dot{\tilde{\gamma}}(t)=\begin{pmatrix}1&0\\0&e^{iyt/n}\,\mathbb{1}\end{pmatrix}\cdot\begin{pmatrix}iy&-e^{iyt}\bar{z}^{\mathrm{t}}\\ze^{-iyt}&-\frac{iy}{n}\,\mathbb{1}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&e^{-iyt/n}\,\mathbb{1}\end{pmatrix}.$$

Hence, $\tilde{\gamma}(t)^{-1}\cdot\dot{\tilde{\gamma}}(t)$ is always perpendicular to the Lie subalgebra $\mathfrak{su}(n)\subset\mathfrak{su}(n+1)$. Since the Riemannian metric on $\mathrm{SU}(n+1)$ is left invariant this means that $\dot{\tilde{\gamma}}(t)$ is perpendicular to the $\mathrm{SU}(n)$-fiber at $\tilde{\gamma}(t)$. $\square$

We recall a simple formula for a map that generates $\pi_{2n}(\mathrm{SU}(n))$ from [PR]. This map

$$\phi:[0,\tfrac{2\pi}{n}]\times\mathbb{S}^{2n-1}\longrightarrow\mathrm{SU}(n)$$

is defined by

$$\phi(\tau,z)=A\cdot\mathrm{diag}(e^{i(n-1)\tau},e^{-i\tau},\ldots,e^{-i\tau})\cdot A^{-1}$$

where $A\in\mathrm{SU}(n)$ is any matrix whose first column is $z\in\mathbb{S}^{2n-1}\subset\mathbb{C}^n$. The values of ϕ are independent of $z\in\mathbb{S}^{2n-1}$ for $\tau=0$ and $\tau=\frac{2\pi}{n}$. Because of this collapse at the endpoints of the interval the map ϕ induces a map $\mathbb{S}^{2n}\to\mathrm{SU}(n)$. More precisely, we now identify the round sphere $\mathbb{S}^{2n}\subset\mathbb{R}\times\mathbb{C}^n$ with the cylinder $[0,\frac{2\pi}{n}]\times\mathbb{S}^{2n-1}$ that is collapsed at the endpoints of the interval by the map

$$\mathbb{S}^{2n}\to\;([0,\tfrac{2\pi}{n}]\times\mathbb{S}^{2n-1})/\!\sim,\qquad \left(\begin{smallmatrix}iy\\z\end{smallmatrix}\right)\mapsto\big(\tfrac{\pi}{n}(y+1),\tfrac{z}{|z|}\big).$$

Theorem 5.2. *We have*

$$\partial_{\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1}}(\mathrm{id})\begin{pmatrix}iy\\z\end{pmatrix}=\begin{pmatrix}-1&0\\0&e^{i\pi/n}\,\mathbb{1}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&\phi\big(\frac{\pi}{n}(y+1),\frac{z}{|z|}\big)\end{pmatrix}.$$

Proof. The following computation shows that ϕ in fact only depends on the first column z of the matrix A:

$$\phi(\tau,z)=e^{-i\tau}\cdot A\cdot\big(\mathbb{1}+\mathrm{diag}(e^{in\tau}-1,0,\ldots,0)\big)\cdot A^{-1}=e^{-i\tau}\big(\mathbb{1}+z(e^{in\tau}-1)\bar{z}^{\mathrm{t}}\big)\,.$$

Straightforward evaluations of $\tilde{\gamma}(\pi)$ and $\phi\big(\frac{\pi}{n}(y+1),\frac{z}{|z|}\big)$ now yield the statement. $\square$

We consider now the special case $n=2$. Recall the following homotopy from [PR]:

$$H_{\mathrm{SU}(3)\to\mathbb{S}^5}(\tau,z,s)=\left(\begin{smallmatrix}z&-\bar{w}\\w&\bar{z}\end{smallmatrix}\right)\begin{pmatrix}e^{i\tau}&0\\0&e^{-i\tau}\end{pmatrix}\left(\begin{smallmatrix}\bar{z}&\bar{w}\\-w&z\end{smallmatrix}\right)\cdot$$
$$\left(\begin{smallmatrix}z&-\bar{w}\\w&\bar{z}\end{smallmatrix}\right)\left(\begin{smallmatrix}\cos s&-\sin s\\\sin s&\cos s\end{smallmatrix}\right)\begin{pmatrix}e^{i\tau}&0\\0&e^{-i\tau}\end{pmatrix}\left(\begin{smallmatrix}\cos s&\sin s\\-\sin s&\cos s\end{smallmatrix}\right)\left(\begin{smallmatrix}\bar{z}&\bar{w}\\-w&z\end{smallmatrix}\right).$$

Theorem 5.3 (see [PR])**.** *The homotopy $H_{\mathrm{SU}(3)\to\mathbb{S}^5}$ deforms ϕ^2 and hence the map*

$$\Big(\left(\begin{smallmatrix}-1&0&0\\0&-i&0\\0&0&-i\end{smallmatrix}\right)\cdot\partial_{\mathrm{SU}(n+1)\to\mathbb{S}^{2n+1}}(\mathrm{id})\Big)^2$$

(for $s=0$) to the constant map to the unit matrix (for $s=\frac{\pi}{2}$).

Remark 5.4. The cases $n > 2$ were also treated in [PR]. Let $\eta : \mathbb{S}^{2n-1} \to \mathrm{SU}(n)$ be a map that generates $\pi_{2n-1}(\mathrm{SU}(n))$ and let η_j denote its j-th column. The n maps $\phi \circ \Sigma\eta_j : \mathbb{S}^{2n} \to \mathrm{SU}(n)$ are all mutually homotopic (by homotopies analogous to the above) and represent the $(n-1)!$-th power of the generator ϕ of $\pi_{2n}(\mathrm{SU}(n))$. The maps $\phi \circ \Sigma\eta_j$ commute mutually and their value-by-value product is the constant map to the identity matrix.

5.2. **The characteristic map of the principal bundle** $\mathrm{Sp}(n+1) \to \mathbb{S}^{4n+3}$**.** The characteristic map of the principal bundle $\mathrm{Sp}(n+1) \to \mathbb{S}^{4n+3}$ generates the homotopy group

$$\pi_{4n+2}(\mathrm{Sp}(n)) \approx \begin{cases} \mathbb{Z}_{(2n+1)!}, & \text{if } n \text{ is even,} \\ \mathbb{Z}_{2(2n+1)!}, & \text{if } n \text{ is odd} \end{cases}$$

(see [Ke2] for the computation of this homotopy group). This follows easily by filling in only stable homotopy groups into the exact homotopy sequence of the bundle. In this subsection we compute the characteristic map using the horizontal lifting construction of section 4.

Let $\mathrm{Sp}(n+1)$ denote the group of all quaternionic $(n+1) \times (n+1)$-matrices A with $\bar{A}^{\mathrm{t}} A = \mathbb{1}$. The group $\mathrm{Sp}(n+1)$ acts transitively on the unit sphere $\mathbb{S}^{4n+3}$ in $\mathbb{H}^{n+1}$. We endow $\mathrm{Sp}(n+1)$ with the up to scaling unique biinvariant Riemannian metric. We consider the curves

$$\gamma(t) = \cos t \,\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \sin t \,\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right)$$

in $\mathbb{S}^{4n+3}$. Here, $p \in \mathrm{Im}\,\mathbb{H}$ is purely imaginary and $u \in \mathbb{H}^n$ is a vector such that $|p|^2 + |u|^2 = 1$. In other words the vector $\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right)$ is contained in the unit sphere $\mathbb{S}^{4n+2}$. For $u \neq 0$ we set

$$\tilde{\gamma}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} - \frac{u}{|u|}\frac{\bar{u}^{\mathrm{t}}}{|u|} \end{pmatrix} + \cos t \begin{pmatrix} 1 & 0 \\ 0 & \frac{u}{|u|} e^{tp} \frac{\bar{u}^{\mathrm{t}}}{|u|} \end{pmatrix} + \sin t \begin{pmatrix} p & -e^{tp}\bar{u}^{\mathrm{t}} \\ u & -\frac{u}{|u|} p e^{tp} \frac{\bar{u}^{\mathrm{t}}}{|u|} \end{pmatrix},$$

where $e^p = \cos|p| + \frac{p}{|p|}\sin|p|$ denotes the exponential map of $\mathbb{S}^3 \subset \mathbb{H}$ from 1.

Lemma 5.5 (see [Du] for $n = 1$)**.** *The curve $\tilde{\gamma}$ is the unique horizontal lift of γ with respect to the projection $\mathrm{Sp}(n+1) \to \mathbb{S}^{4n+3}$ such that $\tilde{\gamma}(0) = \mathbb{1}$. In particular, the above formula for $\tilde{\gamma}$ extends in an analytic way to the case where $u = 0$.*

Proof. It is evident that $\tilde{\gamma}(t) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \gamma(t)$. Hence, $\tilde{\gamma}$ is a lift of γ. A direct computation shows that

$$\tilde{\gamma}(t)^{-1} \cdot \dot{\tilde{\gamma}}(t) = \begin{pmatrix} p & -e^{tp}\bar{u}^{\mathrm{t}} \\ ue^{-tp} & 0 \end{pmatrix}.$$

Hence, $\tilde{\gamma}(t)^{-1} \cdot \dot{\tilde{\gamma}}(t)$ is always perpendicular to the Lie subalgebra $\mathfrak{sp}(n-1) \subset \mathfrak{sp}(n)$. Since the Riemannian metric on $\mathrm{Sp}(n+1)$ is left invariant this means that $\dot{\tilde{\gamma}}(t)$ is perpendicular to the $\mathrm{Sp}(n-1)$-fiber at $\tilde{\gamma}(t)$. $\square$

Note that the fiber of the bundle $\mathrm{Sp}(n+1) \to \mathbb{S}^{4n+3}$ over the south pole $\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$ is $\begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Sp}(n) \end{pmatrix}$. This fiber can be canonically identified with the $\mathrm{Sp}(n)$ fiber over the north pole using left multiplication by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{smallmatrix}\right) \in \mathrm{Sp}(n)$.

Corollary 5.6 (see [Du] for $n = 1$)**.** *The characteristic map* $\partial_{\mathrm{Sp}(n+1)\to\mathbb{S}^{4n+3}}(\mathrm{id}) : \mathbb{S}^{4n+2} \to \mathrm{Sp}(n)$ *is given by*

$$\partial_{\mathrm{Sp}(n+1)\to\mathbb{S}^{4n+3}}(\mathrm{id})\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right) = \mathbb{1} - \tfrac{u}{|u|}(1 + e^{\pi p})\tfrac{\bar{u}^{\mathrm{t}}}{|u|}$$

and hence, for $n = 1$, by

$$\partial_{\mathrm{Sp}(2)\to\mathbb{S}^{7}}(\mathrm{id})\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right) = -\tfrac{u}{|u|}e^{\pi p}\tfrac{\bar{u}}{|u|}.$$

5.3. **The characteristic map of the principal bundle** $\mathrm{G}_2 \to \mathbb{S}^6$**.** In this subsection we identify the characteristic map of the principal bundle $\mathrm{G}_2 \to \mathbb{S}^6$ with the embedding $\eta : \mathbb{S}^5 \to \mathrm{SU}(3)$ given in [PR] (and previously in slightly different form in [CR3] and [Br]).

Let $\mathbb{O} = \mathbb{C} \times \mathbb{C}^3$ denote the octonions with the product introduced in section 3. Octonionic conjugation is given by $\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \bar{z}_0 \\ -z \end{smallmatrix}\right)$. Let $\mathrm{Im}\,\mathbb{O} = i\mathbb{R} \times \mathbb{C}^3$ denote the imaginary octonions. The compact Lie group G_2 is the automorphism group of the octonions. It is a connected subgroup of $\mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ and acts transitively on the sphere $\mathbb{S}^6 \subset \mathrm{Im}\,\mathbb{O}$. Let $\gamma : \mathbb{R} \to \mathbb{S}^6$ be the unit speed geodesic defined by

$$\gamma(t) = \begin{pmatrix} i\cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}.$$

Set

$$e_1(t) = \gamma(t), \quad e_2(t) = \begin{pmatrix} -i\sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix}, \quad e_3(t) = \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \end{pmatrix},$$

$$e_4(2t) = \begin{pmatrix} 0 \\ 0 \\ \cos t \\ i\sin t \end{pmatrix},\ e_5(2t) = \begin{pmatrix} 0 \\ 0 \\ i\cos t \\ \sin t \end{pmatrix},\ e_6(2t) = \begin{pmatrix} 0 \\ 0 \\ -i\sin t \\ \cos t \end{pmatrix},\ e_7(2t) = \begin{pmatrix} 0 \\ 0 \\ -\sin t \\ i\cos t \end{pmatrix}.$$

and define a curve $\tilde{\gamma}(t)$ in $\mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ by mapping $e_k(0)$ to $e_k(t)$ for each k.

Lemma 5.7. *With respect to the fibration* $\mathrm{G}_2 \to \mathbb{S}^6$, $A \mapsto A\big(e_1(0)\big)$, *the curve* $\tilde{\gamma}$ *is the unique horizontal lift of* γ *with* $\tilde{\gamma}(0) = \mathbb{1}$.

Proof. It is easily verified using the octonionic multiplication from section 3 that $e_1(t)\cdot e_2(t) = e_3(t)$, $e_5(t) = e_1(t)\cdot e_4(t)$, $e_6(t) = e_2(t)\cdot e_4(t)$, and $e_7(t) = -e_3(t)\cdot e_4(t)$ for all times t. Hence, the curve $\tilde{\gamma}$ is contained in G_2. Clearly, $\tilde{\gamma}$ is a lift of γ. It is also easily verified that $\tilde{\gamma}$ is a one parameter subgroup of $\mathrm{SO}(\mathrm{Im}\,\mathbb{O})$. Hence, $\tilde{\gamma}$ is a geodesic in G_2 with respect to the up to scaling unique biinvariant metric. It is a standard fact that in order to verify whether a geodesic is horizontal, it suffices to verify horizontality at one point of time. The fiber over $\gamma(0) = \mathbb{1}$ is the natural embedding of $\mathrm{SU}(3)$ into $\mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ as the automorphism group of the complex cross product of section 3. It is straightforward to verify that $\dot{\tilde{\gamma}}(0)$ is perpendicular to this $\mathrm{SU}(3)$ fiber. □

We now recall the embedding $\eta : \mathbb{S}^5 \to \mathrm{SU}(3)$ from [PR] (see also the previous papers [CR3] and [Br]): Let $\mathbb{S}^5$ be the unit sphere in $\mathbb{C}^3$. Set

$$\eta(z) = zz^{\mathrm{t}} + \begin{pmatrix} 0 & -\bar{z}_3 & \bar{z}_2 \\ \bar{z}_3 & 0 & -\bar{z}_1 \\ -z_2 & \bar{z}_1 & 0 \end{pmatrix}.$$

Note that $\eta(Az) = A\eta(z)A^{\mathrm{t}}$ for $A \in \mathrm{SU}(3)$ and that η generates $\pi_5(\mathrm{SU}(3))$. Let $\theta \in \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ denote complex (not octonionic) conjugation on $\mathrm{Im}\,\mathbb{O} = i\mathbb{R} \times \mathbb{C}^3$. The fiber of the bundle $\mathrm{G}_2 \to \mathbb{S}^6$ over $\left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)$ is evidently $\mathrm{SU}(3)\cdot\theta$ since $\theta\left(\begin{smallmatrix}i\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)$.

Theorem 5.8. *We have*

$$\partial_{\mathrm{G}_2\to\mathbb{S}^6}(\mathrm{id})(z) = \eta(-iz)\cdot\theta.$$

Proof. Straightforward evaluation shows $\tilde\gamma(\pi) = \eta\left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)\cdot\theta$. By Lemma 4.10 the characteristic map is equivariant with respect to the transitive action of $\mathrm{SU}(3)$ on $\mathbb{S}^5$ and to the action of $\mathrm{SU}(3)$ on $\mathrm{SU}(3)\cdot\theta$ by conjugation. We have

$$\begin{aligned}\partial_{\mathrm{G}_2\to\mathbb{S}^6}(\mathrm{id})\left(A\cdot\left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)\right) &= A\cdot\eta\left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)\cdot\theta\cdot A^{-1}\\ &= A\cdot\eta\left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)\cdot A^{\mathrm{t}}\cdot\theta = \eta\left(-i\,A\cdot\left(\begin{smallmatrix}-i\\0\end{smallmatrix}\right)\right)\cdot\theta\end{aligned}$$

since $\eta(Az) = A\eta(z)A^{\mathrm{t}}$. □

Remark 5.9. Theorem 5.8 in particular implies the known fact that $\partial_{\mathrm{G}_2\to\mathbb{S}^6}(\mathrm{id})$ generates $\pi_5(\mathrm{SU}(3)) \approx \mathbb{Z}$. This can be used to give an elementary proof of the known fact that $\pi_5(\mathrm{G}_2)$ is trivial (see [Mm]) just by inspecting the relevant part of the exact homotopy sequence of the bundle $\mathrm{G}_2 \to \mathbb{S}^6$.

Remark 5.10. Observe that $\partial_{\mathrm{G}_2\to\mathbb{S}^6}(\mathrm{id})^4 = \mathbb{1}$, while $\partial_{\mathrm{G}_2\to\mathbb{S}^6}(\mathrm{id})$ generates an infinite cyclic homotopy group. This is reflected in the fact that when transfering the boundary map from the fiber over the south pole to the fiber over the north pole the equivariance changes from conjugation to twisted conjugation (twisted by an outer automorphism of $\mathrm{SU}(3)$).

Remark 5.11. It is known (see, e.g., [Ko]) that the action of $\mathrm{SU}(3)\times\mathrm{SU}(3)$ on G_2 by left and right translations is of cohomogeneity one. The geodesic $\tilde\gamma$ is a normal geodesic for this action, i.e., perpendicular to all orbits.

Remark 5.12. The embedding η, i.e., essentially the characteristic map $\partial_{\mathrm{G}_2\to\mathbb{S}^6}(\mathrm{id})$, defines a calibrated submanifold of $\mathrm{SU}(3)$. This result is due to R. Bryant [Br] who classified all codimension 3 cycles of compact Lie groups that are calibrated by the Hodge dual of the fundamental 3-form $(X,Y,Z) \mapsto \langle X,[Y,Z]\rangle$.

5.4. **The characteristic map of the principal bundle** $\mathrm{Spin}(7) \to \mathbb{S}^7$**.** Chaves and Rigas obtained in [CR1] an embedding $\psi : \mathbb{S}^6 \to \mathrm{G}_2$ that genrates $\pi_6(\mathrm{G}_2) \approx \mathbb{Z}^3$. This embedding parametrizes the adjoint orbit of G_2 through one of the elements in the center of the subgroup $\mathrm{SU}(3) \subset \mathrm{G}_2$ and is hence minimal. In this subsection we obtain the embedding ψ by the horizontal lifting construction. This approach is essential for the subsequent proof of Theorem 2.2.

Let $\mathbb{O}$ denote the normed division algebra of the octonions. We use the triality model of $\mathrm{Spin}(8)$:

$$\mathrm{Spin}(8) = \{(A,B,C) \in \mathrm{SO}(\mathbb{O})^3 \mid A(x\cdot y) = B(x)\cdot C(y) \text{ for all } x,y \in \mathbb{O}\}$$

and consider the natural homomorphism $\mathrm{Spin}(8) \to \mathrm{SO}(8)$ given by the projection to the first factor. From this homomorphism we get the standard transitive action

of Spin(8) on the unit octonions $\mathbb{S}^7$. The condition $A(1) = 1$ defines a subgroup $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$, which acts transitively on the unit octonions by

$$\mathrm{Spin}(7) \times \mathbb{S}^7 \to \mathbb{S}^7, \quad (A, B, C) \cdot x = B(x).$$

The isotropy group of $1 \in \mathbb{O}$ with respect to this action is the automorphism group of the octonions

$$\mathrm{G}_2 = \{(A, A, A) \in \mathrm{Spin}(7)\}.$$

From the action of Spin(7) we therefore get the G_2-principal bundle $\mathrm{Spin}(7) \to \mathbb{S}^7$ in the usual way: $(A, B, C) \mapsto (A, B, C) \cdot 1 = B(1)$.

Let γ be a geodesic in the unit octonions with $\gamma(0) = 1$ and $\gamma(\pi) = -1$. In order to give a formula for the horizontal lift of γ to Spin(7) we use the standard notation for the left multiplication, right multiplication, and conjugation on $\mathbb{O}$ with a fixed octonion a:

$$L_a(x) = ax, \quad R_a(x) = xa, \quad C_a(x) = axa^{-1}.$$

Since the octonions form a normed division algebra the transformations L_a, R_a, and C_a are contained in $\mathrm{SO}(\mathbb{O})$.

Lemma 5.13. *The horizontal lift of γ through the unit element of* Spin(7) *is given by*

$$\tilde{\gamma}(3t) = \big(C_{\gamma(t)}, L_{\gamma(t)} \circ R_{\gamma(t)^2}, L_{\bar{\gamma}(t)^2} \circ R_{\bar{\gamma}(t)}\big)\,.$$

Here, $\bar{\gamma}(t)$ is identical to $\gamma(t)^{-1}$ since $\gamma(t)$ is a unit octonion.

Proof. It follows from the Moufang identities

$$L_{aba} = L_a \circ L_b \circ L_a, \quad R_{aba} = R_a \circ R_b \circ R_a$$

that the expression for $\tilde{\gamma}(3t)$ above is contained in Spin(8):

$$a(xy)\bar{a} = a\big(x(a(\bar{a}y))\big)\bar{a} = \big((axa)(\bar{a}y)\big)\bar{a} = \big(\big((axa^2)\bar{a}\big)(\bar{a}y)\big)\bar{a} = (axa^2)(\bar{a}^2y\bar{a}).$$

Moreover, $\tilde{\gamma}(3t)$ is contained in Spin(7) since $C_a(1) = 1$. The formula for $\tilde{\gamma}$ apparently defines a one parameter subgroup of Spin(7) and hence a geodesic. This geodesic projects to the geodesic γ of $\mathbb{S}^7$ since

$$(C_a, L_a \circ R_{a^2}, L_{\bar{a}^2} \circ R_{\bar{a}}) \cdot 1 = a^3.$$

It remains to show that $\tilde{\gamma}$ is horizontal. Since we already know that $\tilde{\gamma}$ is a geodesic, it suffices to verify that $\dot{\tilde{\gamma}}(0)$ is perpendicular to the G_2-fiber over $1 \in \mathbb{S}^7$. For this purpose we note that the inner product of two endomorphisms $Y, Z \in \mathfrak{so}(\mathbb{O}) \times \mathfrak{so}(\mathbb{O}) \times \mathfrak{so}(\mathbb{O})$ is given by $-\operatorname{trace}(Y \circ Z)$. We have $\dot{\gamma}(0) = p \in \mathbb{S}^6 \subset \operatorname{Im}\mathbb{O}$. Hence,

$$\dot{\tilde{\gamma}}(0) = (L_p - R_p, L_p + 2R_p, -2L_p - R_p) \in \mathfrak{so}(\mathbb{O}) \times \mathfrak{so}(\mathbb{O}) \times \mathfrak{so}(\mathbb{O}).$$

The Lie algebra of G_2 consists of endomorphisms of the form $(X, X, X) \in \mathfrak{so}(\mathbb{O}) \times \mathfrak{so}(\mathbb{O}) \times \mathfrak{so}(\mathbb{O})$. It is now obvious that $\operatorname{trace}(\dot{\tilde{\gamma}}(0) \circ (X, X, X))$ vanishes, since the sum of the three components of $\dot{\tilde{\gamma}}(0)$ vanishes. Therefore, $\tilde{\gamma}$ is horizontal. $\square$

Let $\mathbb{S}^6 \subset \mathrm{Im}(\mathbb{O})$ denote the imaginary unit octonions. For any geodesic $\gamma_v(t)$ with $v \in \mathbb{S}^6$ we have $\gamma_v(\pi) = -1$. Thus, $\tilde{\gamma}_v(\pi)$ is contained in the G_2-fiber $\mathrm{Spin}(7)_{-1} \approx \mathrm{G}_2$. Since $(\mathbb{1}, -\mathbb{1}, -\mathbb{1}) \in \mathrm{Spin}(7)$ left translates the north pole $1 \in \mathbb{S}^7$ to the south pole -1 we have $\mathrm{Spin}(7)_{-1} = (\mathbb{1}, -\mathbb{1}, -\mathbb{1}) \cdot \mathrm{G}_2$.

Corollary 5.14. *The characteristic map* $\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id}) : \mathbb{S}^6 \to \mathrm{Spin}(7)_{-1}$ *is given by*

$$\begin{aligned} v \mapsto &\big(C_{\gamma_v(\pi/3)}, L_{\gamma_v(\pi/3)} \circ R_{\gamma_v(2\pi/3)}, L_{\gamma_v(-2\pi/3)} \circ R_{\gamma_v(-\pi/3)}\big) \\ = &(C_{\gamma_v(-2\pi/3)}, -C_{\gamma_v(-2\pi/3)}, -C_{\gamma_v(-2\pi/3)}), \end{aligned}$$

in particular, $\big(\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id})\big)^3 \equiv (\mathbb{1}, -\mathbb{1}, -\mathbb{1})$ *and* $\big(\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id})\big)^6 \equiv (\mathbb{1}, \mathbb{1}, \mathbb{1})$.

Lemma 5.15. *The characteristic map* $\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id}) : \mathbb{S}^6 \to \mathrm{Spin}(7)_{-1}$ *generates the homotopy group* $\pi_6(\mathrm{G}_2) \approx \mathbb{Z}_3$.

Proof. This follows immediately from the exact homotopy sequence of the principal bundle $\mathrm{Spin}(7) \to \mathbb{S}^7$:

$$\mathbb{Z} \approx \pi_7(\mathbb{S}^7) \longrightarrow \pi_6(\mathrm{G}_2) \longrightarrow \pi_6(\mathrm{Spin}(7)) \approx 0. \qquad \square$$

Chaves and Rigas [CR1] obtained the map

$$\psi : \mathbb{S}^6 \to \mathrm{G}_2, \quad p \mapsto (C_{\exp(-2\pi p/3)}, C_{\exp(-2\pi p/3)}, C_{\exp(-2\pi p/3)})$$

which parametrizes the adjoint orbit of G_2 through one of the elements in the center of $\mathrm{SU}(3) \subset \mathrm{G}_2$ and thus represents the homotopy group $\pi_6(\mathrm{G}_2) \approx \mathbb{Z}_3$ in the most geometric way possible: ψ^{3j} is the constant map to the identity matrix in G_2, while the embeddings ψ^{3j+1} and ψ^{3j+2} represent the two nontrivial classes in $\pi_6(\mathrm{G}_2)$.

Theorem 5.16. *We have*

$$\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id}) = (\mathbb{1}, -\mathbb{1}, -\mathbb{1}) \cdot \psi.$$

With this identity we can, of course, also recover from Corollary 5.14 that ψ^3 is the constant map to the identity matrix. Moreover, it follows from Lemma 4.10 that $\partial_{\mathrm{Spin}(7)\to\mathbb{S}^7}(\mathrm{id})$ is G_2-equivariant and hence it is easy to recover that the map ψ indeed parametrizes an adjoint orbit of G_2 through one of the elements in the center of $\mathrm{SU}(3) \subset \mathrm{G}_2$.

5.5. **Proof of Theorem 2.2.** Identify $\mathbb{S}^7$ with the unit sphere in the octonions $\mathbb{O}$ and let $V_{7,2}$ denote the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^7 \approx \mathrm{Im}\,\mathbb{O}$. Let $e_1, e_2, \ldots$ denote an orthonormal basis of $\mathbb{R}^7 \approx \mathrm{Im}\,\mathbb{O}$. Define the map

$$\kappa : \mathbb{S}^7 \to V_{7,2}, \quad a \mapsto (\bar{a}e_1a, \bar{a}e_2a).$$

Identify $\mathrm{Sp}(2)$ with the subgroup of matrices (A, B, C) in $\mathrm{Spin}(7) \subset \mathrm{SO}(\mathbb{O}) \times \mathrm{SO}(\mathbb{O}) \times \mathrm{SO}(\mathbb{O})$, such that, $A \cdot e_1 = e_1$ and $A \cdot e_2 = e_2$, i.e., with $\mathrm{Spin}(5)$.

Let γ_v denote the geodesic of $\mathbb{S}^7$ with $\gamma_v(0) = N$ and $\dot\gamma_v(0) = v$, $|v| = 1$. Apply Lemma 4.6 to the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r] \arrow[d] & \mathrm{G}_2 \arrow[r] \arrow[d] & \mathrm{G}_2/\mathbb{S}^3 \arrow[d, equal] & \\
\mathrm{Sp}(2) \arrow[r] \arrow[d, "\pi"] & \mathrm{Spin}(7) \arrow[r] \arrow[d, "\pi"] & \mathrm{Spin}(7)/\mathrm{Sp}(2) \arrow[r, equal] & V_{7,2} \\
\mathbb{S}^7 \arrow[r, equal] & \mathbb{S}^7 & &
\end{tikzcd}$$

This shows that the curves

$$\tilde\delta_v(t) = \tilde\gamma_v^{\mathrm{Spin}(7)}(t)^{-1} \cdot \tilde\gamma_v^{\mathrm{Sp}(2)}(t)$$

in G_2 are horizontal with respect to the fibration $\mathrm{G}_2 \to V_{7,2}$. The curves $\tilde\delta_v$ satisfy $\tilde\delta_v(0) = \mathbb{1}$ and $\tilde\delta_v(6\pi) = \tilde\gamma_v^{\mathrm{Sp}(2)}(6\pi)$ since $\tilde\gamma_v^{\mathrm{Spin}(7)}(6\pi) = \mathbb{1}$ by Lemma 5.13. The two column vectors of the projected curve $\delta(t)$ in $V_{7,2}$ are given by $\tilde\delta_v(t)\cdot e_1$ and $\tilde\delta_v(t)\cdot e_2$. Here, $\tilde\delta_v(t) \in \mathrm{G}_2$ has three identical entries in $\mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \times \mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \times \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$. Trivially, all of these act identically on e_1 and e_2. In order to evalute the result it is suitable to choose the first component since this is adepted to the embedding of $\mathrm{Sp}(2)$ into $\mathrm{Spin}(7)$. We obtain

$$\tilde\delta_v(3t)\cdot e_1 = \tilde\gamma_v^{\mathrm{Spin}(7)}(3t)^{-1} \cdot \tilde\gamma_v^{\mathrm{Sp}(2)}(3t)\cdot e_1 = \tilde\gamma_v^{\mathrm{Spin}(7)}(3t)^{-1}\cdot e_1 = \overline{\gamma_v(t)}\cdot e_1 \cdot \gamma_v(t)$$

and hence $\tilde\delta_v(6t)\cdot e_1 = \overline{\gamma_v(2t)}\cdot e_1\cdot\gamma_v(2t)$. Similarly, $\tilde\delta_v(6t)\cdot e_2 = \overline{\gamma_v(2t)}\cdot e_2\cdot\gamma_v(2t)$.

Remark 5.17. Toda, Saito, and Yokota proved that the map κ generates $\pi_7(V_{7,2}) \approx \mathbb{Z}_4$. Our construction above yields a direct argument for this fact: The map κ projects to a map $\mathbb{S}^7 \to \mathbb{S}^6$ that generates $\pi_7(\mathbb{S}^6)$ since it can be deformed to the fourth suspension of the Hopf fibration, see subsection 7.1. The claim follows from the relevant part of the exact homotopy sequence of the fibration $\mathbb{S}^5 \cdots V_{7,2} \to \mathbb{S}^6$ (see the introduction).

6. The precise relation between the commutator and the Duran map

In this section we perform Construction 2.1. The domain of definition of the commutator of unit quaternions is $\mathbb{S}^3\times\mathbb{S}^3$. The commutator factors through the smash product $\mathbb{S}^3\wedge\mathbb{S}^3$. It is elementary that the smash product $\mathbb{S}^3\wedge\mathbb{S}^3$ is homeomorphic to $\mathbb{S}^6$ since $\mathbb{S}^3\wedge\mathbb{S}^3$ is the one-point compactification of $\mathbb{R}^3\times\mathbb{R}^3$. For our purposes, however, this homeomorphism is not appropriate, since it does not have any direct relations to the characteristic map $\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})$. Instead, we extract from Duran's explicit formula for $\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})$ a suitable homotopy equivalence between $\mathbb{S}^3\wedge\mathbb{S}^3$ and $\mathbb{S}^6$.

As usual, we define the one point union

$$\mathbb{S}^3\vee\mathbb{S}^3 = \{1\}\times\mathbb{S}^3 \cup \mathbb{S}^3\times\{1\}$$

and the smash product by

$$\mathbb{S}^3 \wedge \mathbb{S}^3 = (\mathbb{S}^3 \times \mathbb{S}^3)/(\mathbb{S}^3 \vee \mathbb{S}^3),$$

The value of the commutator $[a,b]$ is always 1 if $a = 1$ or $b = 1$ thus the commutator $[\,\cdot\,,\,\cdot\,]$ factors through the smash product. Set

$$\iota : \mathbb{S}^6 \to \mathbb{S}^3 \wedge \mathbb{S}^3, \quad \left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right) \mapsto (u/|u|, -e^{\pi p}).$$

Here, $\mathbb{S}^6$ denotes the unit sphere in $\operatorname{Im}\mathbb{H} \times \mathbb{H}$ and e^p denotes the exponential map of $\mathbb{S}^3 \subset \mathbb{H}$ at the point 1 (note that $T_1\mathbb{S}^3 = \operatorname{Im}\mathbb{H}$).

Lemma 6.1. *The map $\iota : \mathbb{S}^6 \to \mathbb{S}^3 \wedge \mathbb{S}^3$ is continuous.*

Proof. The map ι is well-defined since the first $\mathbb{S}^3$-factor collapses if $u = 0$ and hence $|p| = 1$. The smash product $\mathbb{S}^3 \wedge \mathbb{S}^3$ inherits a canonical distance function $\bar{d}$ from the standard metric d on $\mathbb{S}^3 \times \mathbb{S}^3$:

$$\bar{d}([x],[y]) = \min\bigl\{d(x,y), d(x, \mathbb{S}^3 \vee \mathbb{S}^3) + d(y, \mathbb{S}^3 \vee \mathbb{S}^3)\bigr\}.$$

The distance function $\bar{d}$ induces the quotient topology on $\mathbb{S}^3 \wedge \mathbb{S}^3$. Since

$$d(\iota\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right), \mathbb{S}^3 \vee \mathbb{S}^3) \to 0$$

if $u \to 0$ the map ι is continuous. □

It is easy to see that ι is surjective: If $a, b \in \mathbb{S}^3$ with $a \neq 1$ and $b \neq 1$ then the equation

$$\iota\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right) = [(u/|u|, -e^{\pi p})] = [(a,b)]$$

has precisely one solution. If $a = 1$ or $b = 1$ then the solutions of the latter equation are precisely of the form $\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right)$ with $u \in \mathbb{R}$, $u \geq 0$. This is a three dimensional disk D^3 in $\mathbb{S}^6$. The map ι hence factors as follows:

$$\mathbb{S}^6 \longrightarrow \mathbb{S}^6/D^3 \xrightarrow{\lambda} \mathbb{S}^3 \wedge \mathbb{S}^3.$$

The second map λ is continuous and bijective and therefore a homeomorphism. Its inverse is the map

$$\lambda^{-1} : (a,b) \mapsto \left(\begin{smallmatrix} p \\ a\sqrt{1-|p|^2} \end{smallmatrix}\right), \quad \text{where } p \text{ is defined by } b = -e^{\pi p}.$$

The projection map $\mathbb{S}^6 \to \mathbb{S}^6/D^3$ is a homotopy equivalence by standard constructions. Since this essential here, we present the details: Let $f : \mathbb{S}^6 \times [0,1] \to \mathbb{S}^6$ be a homotopy with $f_0 = \mathrm{id}_{\mathbb{S}^6}$, $f_s(D^3) \subset D^3$ and such that $f_1(D^3)$ consists only of one point. An explicit formula for such a homotopy is, for example, given by

$$f_s\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right) = \begin{cases} \left(\begin{smallmatrix} 0 \\ \operatorname{Re} u + s(1+\operatorname{Re} u) \end{smallmatrix}\right) + \sqrt{\frac{1-(\operatorname{Re} u + s(1+\operatorname{Re} u))^2}{1-(\operatorname{Re} u)^2}} \left(\begin{smallmatrix} p \\ \operatorname{Im} u \end{smallmatrix}\right) & \text{for } \operatorname{Re} u \leq \frac{1-s}{1+s}, \\ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) & \text{for } \operatorname{Re} u \geq \frac{1-s}{1+s}. \end{cases}$$

We have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^6 & \xrightarrow{f_t} & \mathbb{S}^6 \\ \downarrow & & \downarrow \\ \mathbb{S}^6/D^3 & \longrightarrow & \mathbb{S}^6/D^3 \end{array}$$

with $f_0 = \mathrm{id}$ and such that for $s = 1$ the induced map $\mathbb{S}^6/D^3 \to \mathbb{S}^6/D^3$ lifts to a map $\hat{f}_1 : \mathbb{S}^6/D^3 \to \mathbb{S}^6$. We have seen:

Lemma 6.2. *The projection map $\mathbb{S}^6 \to \mathbb{S}^6/D^3$ is a homotopy equivalence with homotopy inverse $\hat{f}_1$. Hence, the map $\iota : \mathbb{S}^6 \to \mathbb{S}^3 \wedge \mathbb{S}^3$ is a homotopy equivalence with homotopy inverse $\mu := \hat{f}_1 \circ \lambda^{-1}$.*

Corollary 6.3. *The homotopy*

$$\mathbb{S}^3 \wedge \mathbb{S}^3 \xrightarrow{\lambda^{-1}} \mathbb{S}^6/D^3 \xrightarrow{f_s} \mathbb{S}^6/D^3 \xrightarrow{\lambda} \mathbb{S}^3 \wedge \mathbb{S}^3 \xrightarrow{[\cdot,\cdot]} \mathbb{S}^3$$

is a homotopy between the commutator $[\,\cdot\,,\,\cdot\,]$ and the composition $[\,\cdot\,,\,\cdot\,] \circ \iota \circ \mu$.

Now consider the two homotopies $\mathbb{S}^6 \times [0,1] \to \mathbb{S}^3$,

$$\left(\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right), s\right) \mapsto \tfrac{u}{|u|} e^{\pi p} \tfrac{\bar{u}}{|u|} e^{-(1-s)\pi p + s\pi \mathbf{i}} \quad \text{and} \quad \left(\left(\begin{smallmatrix} p \\ u \end{smallmatrix}\right), s\right) \mapsto \tfrac{u}{|u|} e^{12\pi p} \tfrac{\bar{u}}{|u|} e^{-(1-s)12\pi p}.$$

The first homotopy was given in [DMR], the other is a simple modification.

Lemma 6.4. *The first homotopy deforms $[\,\cdot\,,\,\cdot\,] \circ \iota$ to the characteristic map $\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})$ and hence $[\,\cdot\,,\,\cdot\,]^{12} \circ \iota$ to $\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^{12}$. The second homotopy deforms the composition of the map $(a,b) \mapsto [a, b^{12}]$ and the homotopy equivalence ι to $\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^{12}$.*

Corollary 6.5. *The concatenation of the homotopies in Corollary 6.3 and Lemma 6.4 deform $[\,\cdot\,,\,\cdot\,]^{12}$ and $(a,b) \mapsto [a, b^{12}]$ both to $\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^{12} \circ \mu$ and thus conclude Construction 2.1.*

7. The homotopy between $\kappa \circ \downarrow^2$ and $(N, \Sigma^2\tau)$

In this section we perform Construction 2.3 and construct a homotopy between the map $\kappa \circ \downarrow^2$ (see subsection 5.5) and a map $(N, \Sigma^2\tau)$. Finally, we perform Construction 2.4.

7.1. **The deformation between κ and h.** We first produce a formula for κ using complex coordinates. For this we use the complex cross product definition for the octonionic multiplication on $\mathbb{C} \times \mathbb{C}^3$ from section 3 but not precisely as defined there. For technical purposes we pull it back by the isometry

$$\mathbb{C} \times \mathbb{C}^3 \to \mathbb{C} \times \mathbb{C}^3, \quad \left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \bar{z}_0 \\ iz \end{smallmatrix}\right).$$

This saves us some additional deformations. We also need to specify the basis $e_1, \dots, e_7$ of subsection 5.5. Set

$$e_1 = \begin{pmatrix} i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \end{pmatrix}, \quad \dots, e_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}.$$

Lemma 7.1. *With this convention we have $\kappa\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \left(\kappa_1\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right), \kappa_2\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right)\right)$, where*

$$\kappa_1\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \begin{pmatrix} i(|z_0|^2 - |z|^2) \\ 2\bar{z}_0 z_1 \\ 2\bar{z}_0 z_2 \\ 2\bar{z}_0 z_3 \end{pmatrix}, \qquad \kappa_2\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \begin{pmatrix} -2i \operatorname{Re} z_0 z_1 \\ \bar{z}_0^2 + z_1^2 - |z_2|^2 - |z_3|^2 \\ -2i(\operatorname{Im} z_1) z_2 + 2i(\operatorname{Re} z_0)\bar{z}_3 \\ -2i(\operatorname{Im} z_1) z_3 - 2i(\operatorname{Re} z_0)\bar{z}_2 \end{pmatrix}$$

Proof. Straightforward evaluation of $\left(\begin{smallmatrix} z_0 \\ -iz \end{smallmatrix}\right) \cdot e_1 \cdot \left(\begin{smallmatrix} \bar z_0 \\ iz \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} z_0 \\ -iz \end{smallmatrix}\right) \cdot e_2 \cdot \left(\begin{smallmatrix} \bar z_0 \\ iz \end{smallmatrix}\right)$ with the product defined in Theorem 3.1. □

Remark 7.2. It is evident from the above formulas that the columns of κ are two perpendicular variants of the composition of the fibration $\mathbb{S}^7 \to \mathbb{CP}^3$ and the cut locus collapse $\mathbb{CP}^3 \to \mathbb{S}^6$.

Now omit the argument $\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right)$ for a better readability and set

$$\tilde\kappa_1(s) = \begin{pmatrix} i(|z_0|^2 - |z_1|^2 - (|z_2|^2+|z_3|^2)\cos^2 s) \\ 2\bar z_0 z_1 \\ 2z_2(\sin^2 s + \bar z_0 \cos^2 s) - 2\bar z_3(1-\bar z_0)\cos s \sin s \\ 2z_3(\sin^2 s + \bar z_0 \cos^2 s) + 2\bar z_2(1-\bar z_0)\cos s \sin s \end{pmatrix},$$

$$\tilde\kappa_2(s) = \begin{pmatrix} -2i \operatorname{Re} z_0 z_1 \\ \bar z_0^2 - z_1^2 - (|z_2|^2+|z_3|^2)\cos^2 s \\ 2z_2(\operatorname{Im} z_1)(\cos s \sin s - i\cos^2 s) - 2\bar z_3((1-\operatorname{Re} z_0)\cos s \sin s - i(\sin^2 s + \operatorname{Re} z_0 \cos^2 s)) \\ 2z_3(\operatorname{Im} z_1)(\cos s \sin s - i\cos^2 s) + 2\bar z_2((1-\operatorname{Re} z_0)\cos s \sin s - i(\sin^2 s + \operatorname{Re} z_0 \cos^2 s)) \end{pmatrix}.$$

Keep first κ_2 fixed while deforming κ_1, then keep κ_1 fixed while deforming κ_2, i.e., set

$$H_\kappa(s) = \begin{cases} \big(\tilde\kappa_1(s), \tilde\kappa_2(0)\big), & \text{if } s \in [0, \frac{\pi}{2}], \\ \big(\tilde\kappa_1(\frac{\pi}{2}), \tilde\kappa_2(s - \frac{\pi}{2})\big), & \text{if } s \in [\frac{\pi}{2}, \pi]. \end{cases}$$

Define two perpendicular variants h_1 and h_2 of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ by

$$h_1\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) = \begin{pmatrix} i(|z_0|^2-|z_1|^2) \\ 2\bar z_0 z_1 \end{pmatrix} \text{ and } h_2\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) = \begin{pmatrix} -2i\operatorname{Re} z_0 z_1 \\ \bar z_0^2 - z_1^2 \end{pmatrix}$$

and extend them to all $\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \in \mathbb{C}^2$ by

$$h_1\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) = \left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right| h_1\left(\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) / \left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right|\right) \text{ and } h_2\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) = \left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right| h_2\left(\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) / \left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right|\right).$$

Lemma 7.3. *The homotopy H_κ induces a homotopy $\mathbb{S}^7 \times [0, \pi+1] \to V_{7,2}$ between the map κ and the map h given by*

$$h\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \begin{pmatrix} h_1\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) & h_2\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \\ z_2 & i\bar z_3 \\ z_3 & -i\bar z_2 \end{pmatrix}$$

Proof. First note that $\tilde\kappa_2(s) = f_2 \circ \tilde\kappa_1(s) \circ f_1$, where f_1 and f_2 are the two isometries

$$f_1\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \begin{pmatrix} \operatorname{Re} z_0 + i\operatorname{Im} z_1 \\ \operatorname{Re} z_1 - i\operatorname{Im} z_0 \\ \operatorname{Re} z_2 + i\operatorname{Re} z_3 \\ -\operatorname{Im} z_2 + i\operatorname{Im} z_3 \end{pmatrix} \text{ and } f_2\left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) = \begin{pmatrix} \operatorname{Re} z_0 + i\operatorname{Re} z_1 \\ \operatorname{Im} z_0 + i\operatorname{Im} z_1 \\ \operatorname{Im} z_3 + i\operatorname{Im} z_2 \\ \operatorname{Re} z_3 - i\operatorname{Re} z_2 \end{pmatrix}.$$

Hence, $|\tilde\kappa_2(s)| = |\tilde\kappa_1(s)|$ and a straightforward computation shows

$$|\tilde\kappa_2(s)| = |\tilde\kappa_1(s)| = 1 + (|z_2|^2 + |z_3|^2)\sin^2 s.$$

In this and the following computations one can use the fact that the two vectors $\left(\begin{smallmatrix} z_2 \\ z_3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -\bar z_3 \\ \bar z_2 \end{smallmatrix}\right)$ that arise in the last two components of $\tilde\kappa_1(s)$ and $\tilde\kappa_1(s)$ are perpendicular with respect to the standard hermitian product on $\mathbb{C}^2$. Now we verify first that $\tilde\kappa_1(s)$ and $\tilde\kappa_2(0)$ are linearly independent. If $z_2 = z_3 = 0$ this is clear because h_1 and h_2 are perpendicular. Hence, suppose that $z_2 \neq 0$ or $z_3 \neq 0$. The equation

$$\tilde\kappa_1(s) = \pm\big(1 + (|z_2|^2 + |z_3|^2)\sin^2 s\big)\,\tilde\kappa_2(0)$$

then yields the two equations

$$\begin{aligned}\sin^2 s + \bar z_0 \cos^2 s &= \mp i \operatorname{Im} z_1 \big(1 + (|z_2|^2 + |z_3|^2) \sin^2 s\big),\\ (1-\bar z_0)\cos s \sin s &= \mp i \operatorname{Re} z_0 \big(1 + (|z_2|^2 + |z_3|^2)\sin^2 s\big).\end{aligned}$$

Sorting by real and by imaginary parts one can easily see that there are no solutions. Next we verify that $\tilde\kappa_1(\frac{\pi}{2})$ and $\tilde\kappa_2(s-\frac{\pi}{2})$ are linearly independent. As above this is clear if $z_2 = z_3 = 0$. If $z_2 \neq 0$ or $z_3 \neq 0$ then the equation

$$\big(1 + (|z_2|^2 + |z_3|^2)\sin^2 s\big)\, \tilde\kappa_1(\tfrac{\pi}{2}) = \pm(1 + |z_2|^2 + |z_3|^2)\, \kappa_2(s - \tfrac{\pi}{2})$$

yields the two unsolvable equations

$$\begin{aligned}\big(1 + (|z_2|^2 + |z_3|^2)\sin^2 s\big) &= \pm(1 + |z_2|^2 + |z_3|^2)(\operatorname{Im} z_1)(\cos s \sin s - i\cos^2 s),\\ 0 &= \pm((1 - \operatorname{Re} z_0)\cos s \sin s - i(\sin^2 s + \operatorname{Re} z_0 \cos^2 s)).\end{aligned}$$

Hence, the homotopy H_κ deforms the map κ to the map

$$H_\kappa(\pi) : \mathbb{S}^7 \to V_{7,2}, \quad \left(\begin{smallmatrix} z_0 \\ z \end{smallmatrix}\right) \mapsto \begin{pmatrix} i(|z_0|^2 - |z_1|^2) & -2i \operatorname{Re} z_0 z_1 \\ 2\bar z_0 z_1 & \bar z_0^2 - z_1^2 \\ 2z_2 & 2i\bar z_3 \\ 2z_3 & -2i\bar z_2 \end{pmatrix}$$

and the two columns of $\tilde\kappa$ are linearly independent for all $s \in [0,\pi]$. Now concatenate the homotopy H_κ with the deformation

$$\frac{1}{1 - s + s\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} i(|z_0|^2 - |z_1|^2) & -2i \operatorname{Re} z_0 z_1 \\ 2\bar z_0 z_1 & \bar z_0^2 - z_1^2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + (2 - s)\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ z_2 & i\bar z_3 \\ z_3 & -i\bar z_2 \end{pmatrix}$$

for $s \in [0,1]$ and orthonormalize the two columns for all $s \in [0, \pi + 1]$. □

7.2. **The deformation between $h \circ \downarrow^2$ and $(N, \Sigma^2\tau)$.** Recall first that the map h consists of two perpendicular variants of the fourth suspension $\Sigma^4 h_1$ of the Hopf fibration $h_1 : \mathbb{S}^3 \to \mathbb{S}^2$. Lemma 4.3 provides a concrete null-homotopy of $\Sigma^4 h_1 \circ \downarrow^2$. This null-homotopy can be lifted from $\mathbb{S}^6$ to $V_{7,2}$ and we obtain a homotopy between $h \circ \downarrow^2$ and a map (N, σ) for some map $\sigma : \mathbb{S}^7 \to \mathbb{S}^5$. When we lift the deformation curves horizontally, however, this map σ is not the suspension of a map and we have no idea how to deform $\sigma \circ \downarrow^2$ to a constant map in a subsequent step. In order to obtain a suspension we deform the squaring map of $\mathbb{S}^7$ to the double suspension of the squaring map of $\mathbb{S}^5$ and transfer the problem from $V_{7,2}$ to $V_{5,2}$.

For technical purposes we change our coordinates again. Set $z_2 = a + id$ and $z_3 = c + ib$. More formally, define an isometry $f_3 : \mathbb{R}^4 \to \mathbb{C}^2$ by

$$f_3 : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \left(\begin{smallmatrix} a+id \\ c+ib \end{smallmatrix}\right)$$

and extend it naturally to an isometry $\mathbb{C}^2\times\mathbb{R}^4 \to \mathbb{C}^2\times\mathbb{C}^2$. The map $\tilde{h} = f_3^{-1}\circ h\circ f_3$ is now given by

$$\tilde{h} : \begin{pmatrix} z_0 \\ z_1 \\ a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} h_1\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) & h_2\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \\ a & b \\ b & -a \\ c & -d \\ d & c \end{pmatrix}.$$

Note that $f_3 \circ \downarrow^2 = \downarrow^2 \circ f_3$ since f does not affect the first four real coordinates. Hence, $h \circ \downarrow^2 = f_3 \circ \tilde{h} \circ \downarrow^2 \circ f_3^{-1}$.

Thus it remains to construct a homotopy between $\tilde{h} \circ \downarrow^2$ and $(N, \Sigma^2\tilde{\tau})$ where N is the constant map to the north pole of $\mathbb{S}^7$ and $\tilde{\tau}$ is a map $\mathbb{S}^5 \to \mathbb{S}^3$. The map τ above is then given by $\tau = f_3 \circ \tilde{\tau} \circ f_3^{-1}$.

We now apply a construction that deforms the squaring map such that the last two coordinates remain unaltered.

We consider the squaring map $\downarrow^2$ of the general sphere $\mathbb{S}^n \subset \mathbb{R}\times\mathbb{R}^n$. In Cartesian coordinates this map is given by

$$\downarrow^2 : \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_0^2 - x_1^2 - \ldots - x_n^2 \\ 2x_0x_1 \\ \vdots \\ 2x_0x_n \end{pmatrix}.$$

Given a map $\rho : \mathbb{S}^n \to \mathbb{S}^m$, view $\mathbb{S}^{n+2} \subset (\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}^2$ and $\mathbb{S}^{m+2} \subset (\mathbb{R}\times\mathbb{R}^m)\times\mathbb{R}^2$ as double lower suspensions of $\mathbb{S}^n$ and $\mathbb{S}^m$ and let

$$\Sigma_2\rho : \mathbb{S}^{n+2} \to \mathbb{S}^{m+2}, \quad \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} |v|\rho(v/|v|) \\ w \end{smallmatrix}\right)$$

denote the double lower suspension of ρ. For $x \in \mathbb{S}^{n+2}$ and $s \in [0, \frac{\pi}{2}]$ set

$$H_{\downarrow^2}(x,s) = \begin{pmatrix} x_0^2 - x_1^2 - \ldots - x_n^2 - (x_{n+1}^2 + x_{n+2}^2)\cos^2 s) \\ 2\bar{x}_0x_1 \\ \vdots \\ 2\bar{x}_0x_n \\ 2x_{n+1}(\sin^2 s + x_0\cos^2 s) - 2x_{n+2}(1-x_0)\cos s\sin s \\ 2x_{n+2}(\sin^2 s + x_0\cos^2 s) + 2x_{n+1}(1-x_0)\cos s\sin s \end{pmatrix}.$$

Lemma 7.4. *The homotopy $H_{\downarrow^2}$ induces a homotopy $\mathbb{S}^{n+2}\times[0,\frac{\pi}{2}] \to \mathbb{S}^{n+2}$ between the squaring map $\downarrow^2$ of $\mathbb{S}^{n+2}$ and the double lower suspension $\Sigma_2\downarrow^2$ of the squaring map of $\mathbb{S}^n$.*

Proof. Contained in the proof of Lemma 7.3. □

Thus it remains to construct a homotopy between $\tilde{h}\circ\Sigma_2\downarrow^2$ and $(N, \Sigma^2\tilde{\tau})$.

We now apply the following construction: Let

$$(\alpha, \beta) : \mathbb{S}^n \times [0,1] \to V_{m,2} \subset \mathbb{S}^{m-1}\times\mathbb{S}^{m-1}$$

be a homotopy between the two maps (α_0, β_0) and (α_1, β_1). Extend α and β to maps $\mathbb{R}^{n+1}\times[0,1] \to \mathbb{R}^m$ by setting $\alpha(v,s) = |v|\alpha(\frac{v}{|v|}, s)$ and $\beta(v,s) = |v|\beta(\frac{v}{|v|}, s)$.

Now define the two homotopies $\tilde\alpha, \tilde\beta : \mathbb{S}^{n+2} \times [0,1] \to \mathbb{S}^{m+1} \subset \mathbb{R}^{m+2}$ by setting

$$\tilde\alpha(\left(\begin{smallmatrix} v \\ c \\ d \end{smallmatrix}\right), s) = \left(\begin{smallmatrix} \alpha(v,s) \\ c \\ d \end{smallmatrix}\right) \quad \text{and} \quad \tilde\beta(\left(\begin{smallmatrix} v \\ c \\ d \end{smallmatrix}\right), s) = \left(\begin{smallmatrix} \beta(v,s) \\ -d \\ c \end{smallmatrix}\right).$$

It is evident that $\tilde\alpha$ and $\tilde\beta$ are always perpendicular. The following statement is now obvious.

Lemma 7.5. *The assignment $(\alpha, \beta) \mapsto (\tilde\alpha, \tilde\beta)$ induces a homomorphism $\pi_n(V_{m,2}) \to \pi_{n+2}(V_{m+2,2})$.*

Thus it remains to construct a homotopy between $\tilde h_{5,2} \circ \downarrow^2$ and the map $(N, \tilde\tau)$, where $\tilde h_{5,2} : \mathbb{S}^5 \to V_{5,2}$ is the map

$$\tilde h_{5,2} : \left(\begin{smallmatrix} z_0 \\ z_1 \\ a \\ b \end{smallmatrix}\right) \mapsto \begin{pmatrix} h_1\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) & h_2\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \\ a & b \\ b & -a \end{pmatrix}.$$

This is because the concrete homomorphism of Lemma 7.5 yields a deformation

$$\tilde h \circ \Sigma_2 \downarrow^2 \begin{pmatrix} z_0 \\ z_1 \\ a \\ b \\ c \\ d \end{pmatrix} \sim \begin{pmatrix} i\left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right| & 0 \\ 0 & \tilde\tau\begin{pmatrix} z_0 \\ z_1 \\ a \\ b \end{pmatrix} \\ c & -d \\ d & c \end{pmatrix}$$

and the latter map can be deformed by

$$\begin{pmatrix} i\cos(s\arccos(\left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right|)) & 0 \\ 0 & \tilde\tau \\ \frac{c}{\sqrt{a^2+b^2+c^2+d^2}}\sin(s\arccos(\left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right|)) & -d \\ \frac{d}{\sqrt{a^2+b^2+c^2+d^2}}\sin(s\arccos(\left|\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right)\right|)) & c \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} i & 0 \\ 0 & \tilde\tau \\ 0 & -d \\ 0 & c \end{pmatrix}.$$

and, finally, by applying some evident rotations in the domain of definition and in the target domain, to the map $(N, \Sigma^2\tilde\tau)$.

In the rest of this subsection we construct the remaining homotopy between $\tilde h_{5,2} \circ \downarrow^2$ and $(N, \tilde\tau)$. It is easy to deform the identity on $\mathbb{C}^2 \times \mathbb{R}^2$ to the isometry

$$f_5 : \left(\begin{smallmatrix} z_0 \\ z_1 \\ a \\ b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a+ib \\ z_1 \\ \operatorname{Re} z_0 \\ \operatorname{Im} z_0 \end{smallmatrix}\right)$$

and the identity on $i\mathbb{R} \times \mathbb{C} \times \mathbb{R}^2$ to the isometry

$$f_6 : \left(\begin{smallmatrix} iy \\ w \\ a \\ b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} ia \\ b+iy \\ \operatorname{Re} w \\ \operatorname{Im} w \end{smallmatrix}\right).$$

The map $\tilde h_{5,2}$ is thus homotopic to the map $\hat h_{5,2} = f_6 \circ \tilde h_{5,2} \circ f_5$ and, accordingly, $\tilde h_{5,2} \circ \downarrow^2$ is homotopic to $\hat h_{5,2} \circ \downarrow^2$. The first column of the map $\hat h_{5,2}$ is now nothing but the second suspension $\Sigma^2 h_1$ of h_1. By Lemma 4.2 we have

$$\Sigma^2 h_1 \circ \downarrow^2 = \downarrow^2 \circ \Sigma^2 h_1$$

and an explicit null-homotopy of $\downarrow^2 \circ \Sigma^2 h_1$ is supplied in Lemma 4.3. Starting with our map $\hat{h}_{5,2} \circ \downarrow^2$ this null-homotopy $\mathbb{S}^5 \times [0,1] \to \mathbb{S}^4$ can be lifted horizontally to the Stiefel manifold $V_{5,2}$. This yields a homotopy

$$\hat{h}_{5,2} \circ \downarrow^2 \sim \begin{pmatrix} i & 0 \\ 0 & \tilde{\tau} \end{pmatrix}$$

with a map $\tilde{\tau} : \mathbb{S}^5 \to \mathbb{S}^3$. It is not difficult to solve the ODE $\delta'(s) = -\langle \delta(s), \gamma'(s) \rangle \gamma(s)$ for the horizontal lift $(\gamma(s), \delta(s))$ of the curves $\gamma(s) = H_1(\ldots, s)$ and $\gamma(s) = H_2(\ldots, s)$ of Lemma 4.3 explicitly and thus to write down a closed formula for the map $\tilde{\tau}$. We omit to present this lengthy formula since it is completely irrelevant for the following steps.

7.3. **Performance of Construction 2.4.** Now let α_s denote the deformation between the two maps $\alpha_0 = \kappa \circ \downarrow^2$ and $\alpha_1 = (N, \Sigma^2 \tau)$ constructed in the previous two subsections. Note that the image of the south pole $\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix} \right)$ under α_s varies with s. Thus, the deformation $\partial_{\mathrm{G}_2 \to V_{7,2}}(\alpha_s)$ does not take place in just one $\mathbb{S}^3$-fiber. There are several elementary ways to fix this issue. We choose one that fits perfectly to the subsequent homotopie $H_{\mathrm{SU}(3) \to \mathbb{S}^5}$ given in Theorem 5.3. Let $v_0 = \left(\begin{smallmatrix} i \\ 0 \end{smallmatrix} \right) \in \mathbb{S}^7 \subset \mathbb{C} \times \mathbb{C}^3$ and $\gamma_{v_0}(t) = \left(\begin{smallmatrix} e^{it} \\ 0 \end{smallmatrix} \right)$. By Lemma 7.1 we have

$$\alpha_0 \circ \gamma_{v_0}(t) = \left(\begin{smallmatrix} i & 0 \\ 0 & e^{-2it} \\ 0 & 0 \\ 0 & 0 \end{smallmatrix} \right)$$

and, by the definition of the suspension, we have

$$\alpha_1 \circ \gamma_{v_0}(t) = \left(\begin{smallmatrix} i & 0 \\ 0 & e^{it} \\ 0 & 0 \\ 0 & 0 \end{smallmatrix} \right).$$

In both cases, the horizontal lifts of these curves are contained in $\mathrm{SU}(3) \subset \mathrm{G}_2$, since the first column is constant. It is easily verified that

$$\widetilde{\alpha_0 \circ \gamma_{v_0}}(t) = \left(\begin{smallmatrix} e^{-2it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{it} \end{smallmatrix} \right)$$

and

$$\widetilde{\alpha_1 \circ \gamma_{v_0}}(t) = \left(\begin{smallmatrix} e^{it} & 0 & 0 \\ 0 & e^{-it/2} & 0 \\ 0 & 0 & e^{-it/2} \end{smallmatrix} \right).$$

Hence,

$$\partial_{\mathrm{G}_2 \to V_{7,2}}(\alpha_0)(v_0) = \widetilde{\alpha_0 \circ \gamma_{v_0}}(\pi) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \right)$$

and

$$\partial_{\mathrm{G}_2 \to V_{7,2}}(\alpha_1)(v_0) = \widetilde{\alpha_1 \circ \gamma_{v_0}}(\pi) = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{smallmatrix} \right).$$

We now consider the homotopy

$$\partial_{\mathrm{G}_2 \to V_{7,2}}(\alpha_0)(v_0) \cdot \left(\partial_{\mathrm{G}_2 \to V_{7,2}}(\alpha_s)(v_0) \right)^{-1} \cdot \partial_{\mathrm{G}_2 \to V_{7,2}}(\alpha_s).$$

This homotopy deforms $\partial_{\mathrm{G}_2 \to V_{7,2}}(\kappa \circ \downarrow^2)$ to

$$\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{smallmatrix} \right) \cdot \partial_{\mathrm{G}_2 \to V_{7,2}}(N, \Sigma^2 \tau) = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{smallmatrix} \right) \cdot \partial_{\mathrm{SU}(3) \to \mathbb{S}^5}(\Sigma \tau)$$

within the fixed $\mathbb{S}^3$-fiber over $(e_1, e_2) \in V_{7,2}$. The last map fits perfectly to Theorem 5.3.

8. Nontrivial maps from $\mathbb{S}^7$ to $\mathrm{Sp}(2)$ and exotic actions on $\mathbb{S}^7 \times \mathbb{S}^3$

8.1. **Nontrivial maps from $\mathbb{S}^7$ to $\mathrm{Sp}(2)$.** Consider $\mathbb{S}^7$ with north pole $N = (1, 0, \ldots, 0)$. As in section 4 let γ_v denote the geodesic of $\mathbb{S}^7$ (with respect to the standard metric on $\mathbb{S}^7$) with $\gamma_v(0) = N$ and $\dot\gamma_v(0) = v$. Let $D^7(\pi)$ denote the disk in the tangent space $T_N\mathbb{S}^7$ with radius π. For a fixed but arbitrary $j \in \mathbb{Z}$ lift the geodesics $t \mapsto \gamma_v(12jt)$ horizontally with respect to the fibration $\mathrm{Sp}(2) \to \mathbb{S}^7$. This yields a map $\xi_j : D^7(\pi) \to \mathrm{Sp}(2)$ with

$$\xi_j\big|_{\mathbb{S}^6(\pi)} = \partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\downarrow^{12j}).$$

On the other hand, the null-homotopy of

$$\big(\partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\mathrm{id})\big)^{12j} = \partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\downarrow^{12j})$$

(see Corollary 4.9 for this identity) provides us with a map $\zeta_j : D^7(\pi) \to \mathbb{S}^3$ with

$$\zeta_j\big|_{\mathbb{S}^6(\pi)} = \partial_{\mathrm{Sp}(2)\to\mathbb{S}^7}(\downarrow^{12j}).$$

We now define a map $\chi_j : D^7(\pi) \to \mathrm{Sp}(2)$ by

$$\chi_j(v) = \xi_j(v) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & \zeta_j^{-1} \end{smallmatrix}\right).$$

Lemma 8.1. *The map χ_j induces a map $\mathbb{S}^7 \to \mathrm{Sp}(2)$ whose first column is the $12j$-th power $\downarrow^{12j}$ of $\mathbb{S}^7$. Hence, the map χ_j represents the j-th homotopy class in $\pi_7(\mathrm{Sp}(2)) \approx \mathbb{Z}$.*

Proof. By definition the map χ_j evaluates constantly to the north pole on the boundary of $D^7(\pi)$. Hence, we get a map from $\mathbb{S}^7$ to $\mathrm{Sp}(2)$ whose first column is $\downarrow^{12j}$, a map of degree $12j$. The last claim follows now from the relevant part of the exact homotopy sequence of the bundle $\mathrm{Sp}(2) \to \mathbb{S}^7$. □

8.2. **Exotic actions on $\mathbb{S}^7 \times \mathbb{S}^3$.** We now combine the maps χ_j above with the generalized Gromoll-Meyer construction in [DPR]. Let E^{10}_n denote the $\mathbb{S}^3$-principal bundle obtained by pulling back $\mathrm{Sp}(2) \to \mathbb{S}^7$ by the n-th power $\downarrow^n$ of $\mathbb{S}^7$:

$$\begin{array}{ccc} E^{10}_n & \longrightarrow & \mathrm{Sp}(2) \\ \big\downarrow & & \big\downarrow \\ \mathbb{S}^7 & \xrightarrow{\ \downarrow^n\ } & \mathbb{S}^7. \end{array}$$

Explicitly,

$$E^{10}_n := \big\{(u, v) \in \mathbb{S}^7 \times \mathbb{S}^7 \;\big|\; \langle\!\langle \downarrow^n(u), v\rangle\!\rangle = 0\big\}$$

where $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ denotes the standard Hermitian inner product on the quaternionic vector space $\mathbb{H}^2$. The total spaces E^{10}_n come equipped with a free action of the unit quaternions:

$$\mathbb{S}^3 \times E^{10}_n \to E^{10}_n, \quad q \star (u, v) = (qu\bar q, qv).$$

Here, $qu\bar{q}$ means that the two quaternionic components of u are simultaneously conjugated by $q \in \mathbb{S}^3$. The quotient of E_n^{10} by the free $\star$-action is a smooth manifold

$$\Sigma_n^7 := E_n^{10}/\mathbb{S}^3.$$

Theorem 8.2 ([DPR]). *The differentiable manifold Σ_n^7 is a homotopy sphere and represents the $(n \mod 28)$-th element in $\Theta_7 \approx \mathbb{Z}_{28}$.*

Now the map $\chi_j : \mathbb{S}^7 \to \mathrm{Sp}(2)$ from the previous subsection supplies us immediately with the section $\mathbb{S}^7 \to E_{12j}^{10}$, $u \mapsto (u, \chi_{j,2}(u))$ of the principal bundle $E_{12j}^{10} \to \mathbb{S}^7$. Here, $\chi_{j,2}(u)$ means the second column of χ_j. We obtain the trivialization

$$\mathbb{S}^7 \times \mathbb{S}^3 \to E_{12j}^{10}, \quad (u, r) \mapsto \big(u, \chi_{j,2}(u) r\big)$$

with inverse

$$E_{12j}^{10} \to \mathbb{S}^7 \times \mathbb{S}^3, \quad (u, v) \mapsto \big(u, \langle\!\langle \chi_{j,2}(u), v \rangle\!\rangle\big).$$

This trivialization can be used to transfer the Gromoll-Meyer action from E_{12j}^{10} to the product $\mathbb{S}^7 \times \mathbb{S}^3$. This way we obtain the formula of Theorem 1.2 from the introduction.

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References

[AC] M. Arkowitz, C.R. Curjel, *Some properties of the exotic multiplications on the three-sphere*, Quart. J. Math. **20** (1969), 171–176.

[BR] T. E. Barros, A. Rigas, *The role of commutators in a non-cancellation phenomenon*, Math. J. Okayama Univ. **43** (2001), 73–93.

[BH] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces. II.*, Amer. J. Math. **81** (1959), 315–382.

[BS] A. Borel, J.-P. Serre, *Groupes de Lie et puissances reduites de Steenrod*, Amer. J. Math. **75**, (1953). 409–448.

[Bt] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.

[Bd] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics 139, Springer, New York, 1993.

[Br] R. Bryant, *Minimizing cycles of codimension 3 in compact simple Lie groups*, preprint 2001.

[CR1] L.M. Chaves, A. Rigas, *On a conjugate orbit of G_2*, Math. J. Okayama Univ. **33** (1991), 155–161.

[CR2] L.M. Chaves, A. Rigas, *Hopf maps and triality*, Math. J. Okayama Univ. **38** (1996), 197–208.

[CR3] L.M. Chaves, A. Rigas, *Complex reflections and polynomial generators of homotopy groups*, J. Lie Theory **6** (1996), 19–22.

[Du] C. E. Duran, *Pointed Wiedersehen metrics on exotic spheres and diffeomorphisms of* $\mathbb{S}^6$, Geom. Dedicata **88** (2001), 199–210.

[DMR] C. E. Duran, A. Mendoza, A. Rigas, *Blakers-Massey elements and exotic diffeomorphisms of* S^6 *and* S^{14}, Trans. Amer. Math. Soc. **356** (2004), 5025–5043.

[DPR] C. E. Duran, T. Püttmann, A. Rigas, *An infinte family of Gromoll-Meyer spheres*, Arch. Math. **95** (2010), 269–282.

[HR] P. James, J. Roitberg, *Note on principal* S^3*-bundles*, Bull. Amer. Math. Soc. **74** (1968), 957–959.

[Hu] S. T. Hu, *Homotopy theory*, Pure and Applied Mathematics VIII, Academic Press, New York, 1959.

[Ja] I. M. James, *On* H*-spaces and their homotopy groups*, Quart. J. Math. Oxford Ser. (2) **11** (1960), 161–179.

[Ke1] M. Kervaire, *Some nonstable homotopy groups of Lie groups*, Illinois J. Math. **4** (1960), 161–169.

[Ke2] M. Kervaire, *On the Pontryagin classes of certain* $\mathrm{SO}(n)$*-bundles over manifolds*, Amer. J. Math. **80** (1958), 632–638.

[Ko] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571-612.

[Mm] M. Mimura, *The homotopy groups of Lie groups of low rank*, J. Math. Kyoto Univ. **6** (1967), 131–176.

[Pü1] T. Püttmann, *Einige Homotopiegruppen der klassischen Gruppen aus geometrischer Sicht*, Habilitationsschrift, Ruhr-Universität Bochum 2004.

[Pü2] T. Püttmann, *Cohomogeneity one manifolds and self-maps of nontrivial degree*, Transf. Groups **14** (2009), 225–247.

[PR] T. Püttmann, A. Rigas, *Presentations of the first homotopy groups of the unitary groups*, Comment. Math. Helv. **78** (2003), 648–662.

[Ri] A. Rigas, $\mathbb{S}^3$*-bundles and exotic actions*, Bull. Soc. Math. France **112** (1984), 69–92.

[Ro] V. A. Rokhlin, *Classification of mappings of an* $(n+3)$*-dimensional sphere into an* n*-dimensional one*, Doklady Akad. Nauk SSSR **81** (1951), 19–22.

[Sa] H. Samelson, *Groups and spaces of loops*, Comment. Math. Helv. **28** (1954), 278–287.

[St] N. E. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press 1951.

[TSY] H. Toda, Y. Saito, T. Yokota, *A note on the generator of* $\pi_7(\mathrm{SO}(n))$, Mem. Coll. Sci. Uni. Kioto Ser. A **30** (1957), 227–230.

[Wh] G. W. Whitehead, *On mappings into group-like spaces*, Comment. Math. Helv. **28**, 320–328.

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